Revisiting Fluxions

A Kinematic Foundation for Calculus

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May 10, 2026

Document Version 1.00

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Notation

Logic

※	Contradiction (in proofs).
$\neg p$	Not p, where p is a proposition.
$p \wedge q$	The conjunction of both p and q holds.
$p \vee q$	At least one of propositions p or q holds (perhaps both do).
$p \Rightarrow q$	Proposition p implies q.
p iff q	Proposition p holds if and only if q holds, ie. p is logically equivalent to q.
$P(x)$	The proposition that predicate P is true of object x.
$\exists x\, P(x)$	There exists an x for which $P(x)$ is true.
$\forall x\, P(x)$	For all x, $P(x)$ is true.
$X \equiv Y$	Formula X is equivalent to (or defined as) Y.

A proposition is a statement that is either true or false.

Naively, a unary predicate P on a class of objects X is a function of $x \in X$ taking a value in the set $\{\text{true}, \text{false}\}$. One can think of it as a $\{\text{true}, \text{false}\}$ property of $x \in X$. In places where one expects a proposition, the term $P(x)$ represents the proposition that predicate P holds true of x.

Sets

$x \in S$	x is a member of set S.
$S \subseteq T$	Set S is a subset of set T (it may equal T).
$\{x \in S \mid P(x)\}$	The set of x in a possibly larger set S satisfying property P.
$S \cap T$	The intersection (common elements) of sets S and T, where $S \cap T = \{x \mid x \in S \text{ and } x \in T\}$.
$S \cup T$	The union of sets S and T (elements in at least one of the sets S or T), so that $S \cup T = \{x \mid x \in S \text{ or } x \in T\}$.
$\mathbb{N}$	The set of natural numbers, 0, 1, 2, etc.
$\mathbb{Z}$	The set of integers, 0, ± 1, ± 2, etc.
$\mathbb{Q}$	The set of rational numbers $\frac{p}{q}$, for $p, q \in \mathbb{Z}$ with $q \neq 0$.

$\mathbb{R}$	The set of real numbers.
$\mathbb{R} \cup \{\pm\infty\}$	The set of extended real numbers, which includes $\pm\infty$.
$\inf S$	Infimum (greatest lower bound) of a set S; usually $S \subseteq \mathbb{R}$.
$\sup S$	Supremum (least upper bound) of a set S; usually $S \subseteq \mathbb{R}$.

Functions

$f(x)$	A function f of variable x; or the value of f at x.
$f^{-1}S$ or $f^{-1}(S)$	The pre-image of a set S, ie. $f^{-1}S = \{x \mid f(x) \in S\}$. Eg. $f^{-1}\{y\}$ is the pre-image of a value y.
$f\|_S$	Function f restricted to the sub-domain S.
$f \circ g$	The composition of functions f and g. Function $h = f \circ g$ iff $h(x) = f(g(x))$.

Intervals

$]a, b[$	Open interval, ie. $x \in \mathbb{R}$ with $a < x < b$.
$[a, b]$	Closed interval, with $a \leq x \leq b$.
$]a, b]$	Half (ie. left) open interval, with $a < x \leq b$.
$[a, b[$	Half (ie. right) open interval, with $a \leq x < b$.
I^o	Interior (hence largest open sub-interval) of interval I. If $I = [a, b]$, then $I^o =]a, b[$.
ΔI	Length of (a usually finite) interval I.
$\min I$, $\max I$	Minimum and maximum endpoints of a *closed* interval I. If $I = [a, b]$, $\min I = a$ and $\max I = b$
$\operatorname{opp} I$	The opposite of an interval I. If $I = [a, b]$, then $\operatorname{opp} I = [-b, -a]$.

Symbols for Intervals

Non-trivial closed intervals play a significant role in the present fluent theory, much as open intervals do in classical analysis. This document tries to follow a consistent naming convention. All the following symbols refer to intervals which are non-trivial, finite and closed.

G	Test interval. Usually, $G \subseteq H$ or $G \subseteq J$.
H	Restricted indexing interval $H \subseteq I$, hence $x\|_H$. Eg. $H = I^\dagger \cap I^\ddagger$ in a unified two-sided flow comparison. Also used in a relabelling.
I	Primary indexing interval for a fluent.
$I^\dagger$	Local interval in one-sided flow velocity comparisons. Eg. $u\psi_1 \lessdot x$ on $I^\dagger$ through $t_0 \in (I^\dagger)^o$.

$I^\ddagger$	Local interval in one-sided flow velocity comparisons. Eg. $x \lessdot v\psi_1$ on $I^\ddagger$ through $t_0 \in (I^\ddagger)^o$.
J	Test interval. Usually, $J \subseteq I$.
K	Primary indexing interval for a fluent. Often $I \subseteq K$, where I and K are indexing intervals of essentially the same fluent.
$r\colon H \to I$	Relabelling r of I by H.

Often:

$$\begin{aligned} t \in I &= [T_1, T_2], \\ J &= [t_1, t_2], \\ t_0 \in J^o &\subseteq I, \end{aligned}$$

where $t, t_0, t_1, t_2, T_1, T_2 \in \mathbb{R}$.

Fluents

$x[t]$	Fluent x parametrised by t; or the value of x at t.
$\operatorname{opp} x$	The opposite of a fluent x. If fluent $y = \operatorname{opp} x$, then $y[s] = -x[t]$, where $s = -t$ for each t in the indexing interval of x.
$x \circ r$	The fluent x relabelled by the function r. When the fluent $y = x \circ r$ and time-like instant $t = r(s)$, then $y[s] \equiv (x \circ r)[s] = x[t]$, sometimes written simply as $x[s]$.
$x\|_J$	Fluent x restricted to an indexing interval J.
$\Delta x\|_J$	Change in fluent x across indexing interval J.
$x \lessdot y$	Fluent x flows slower than fluent y (eg. on a non-trivial closed interval I; or through a time-like instant t_0).
$\acute{x}$	Greatest lower flow velocity bound (glfv) of fluent x (eg. through a time-like instant t_0).
$\grave{x}$	Least upper flow velocity bound (lufv) of fluent x.
$\dot{x}$ or $\operatorname{flx} x$	Unique flow velocity limit of fluent x, or the derived fluxion, if either exist.
$[\partial x \mid \partial t]$	Flow velocity interval $[\grave{x}, \acute{x}]$ of a fluent $x[t]$ with respect to its time-like parameter t.

See Chapters 3, 4, 5 and 6.

Symbols for Fluents

ψ_1	The fluent continuation $\psi_1[t] = t$ for each $t \in \mathbb{R}$.

See Chapter 5.

Fluxions

$\dot{x}[t]$	Fluxion of fluent x with respect to parameter t; or the value of the fluxion $\dot{x}$ at t.
$\ddot{x}$	Fluxion of the fluxion of a fluent x.

See Chapter 7.

Preface

Newton on Quadrature

In his *Introduction to the Quadrature of Curves* (originally *Introductio ad Quadraturam Curvarum*), Isaac Newton wrote:

“I don’t here consider Mathematical Quantities as composed of Parts extreamly small, but as generated by a continual motion. Lines are described, and by describing are generated, not by any apposition of Parts, but by a continual motion of Points. Surfaces are generated by the motion of Lines, Solids by the motion of Surfaces, Angles by the Rotation of their Legs, Time by a continual flux, and so in the rest. These Geneses are founded upon Nature, and are every Day seen in the motion of Bodies.

“And after this manner the Ancients by carrying moveable right Lines along immoveable ones in a Normal Position or Situation, have taught us the Geneses of Rectangles.

“Therefore considering that Quantities, encreasing in equal times, and generated by this encreasing, are greater or less, according as their Velocity by which they encrease, and are generated, is greater or less; I endeavoured after a Method of determining the Quantities from the Velocities of their Motions or Increments, by which they are generated; and by calling the Velocities of the Motions, or of the Augments, by the Name of Fluxions, and the generated Quantities Fluents, I (in the years 1665 and 1666) did, by degrees, light upon the Method of Fluxions, which I here make use of in the Quadrature of Curves.”

(From the English translation by John Harris.)

Aim

These notes provide a foundation for differential calculus, loosely inspired by Isaac Newton’s fluents and fluxions. The aim is to add to existing modern approaches by exhibiting a method that draws on our kinematic intuitions and is logically rigorous. It’s based on what it might mean for one object to be going faster than another. The key intuition is one which is very easy to grasp: that the faster you go, the further you travel in a given time.

The ideas expressed here manifest various interwoven personal thoughts about the foundations of analysis (and mathematics more generally), especially about the interplay between logic and intuition. This isn't intended in any way as a substitute text book on analysis; nor does it pretend to any worthwhile historical insight in the way it evokes Newton. Instead, I simply found much to intrigue me when exploring the implications of this approach, and wanted to meet the challenge of presenting a coherent account of what seemed to be the main results.

Different Perspectives

Isaac Newton produced his own form of calculus during the Great Plague of 1665/66. His approach to quadrature gave us integral calculus, which systematised the calculation of the area of curves in plane geometry. Building on earlier work, he recognised differential calculus as the inverse process, which then provided the necessary mathematical machinery for his Laws of Motion. Although the terms he coined seem alien to us today, they suggest the image of a value flowing through time. He used the method of *fluents* and *fluxions*, where nowadays we might refer to variables and their rates of change. This document recycles his terminology and very loosely borrows from his ideas.

The result is a kinematic foundation for differential calculus, appealing to quantities in flow and the concept of one quantity flowing faster than another. It tries to answer a nagging personal dissatisfaction with our current established presentations. Historically, these might appeal to infinitesimally small increments or informal ideas about numeric or geometric limits. Eventually, nineteenth century mathematicians developed a more rigorous arithmetical foundation based on the ε-δ definition of a limit (commonly employing the Greek letters epsilon and delta for inter-linked tolerances, which are positive in value and thought of as arbitrarily small). Yet, while their approach achieved the rigour they needed, it seems to lose any sense that change is inherently dynamic.

In the approach described here, I've tried to recapture an intuition of variation occurring within the continuous progress of time. In modern analysis, continuous geometric objects and motions tend to be described as (infinitely large) sets of individual points. There's perhaps an analogy with chemical analysis, where we might compare points to the indivisible atoms of a chemical element. Just as a chemist analyses which elements make up the material they're studying to determine the molecular bonds that hold it together, a modern mathematical analyst will appeal to the relations between nearby points in the space that's absorbing their attention. They atomise the space conceptually, only to then stick it back together again.

Although I'll often try to avoid treating continuous entities as infinite collections of points, set-theoretic methods are both powerful and well-proven tools in point-based analysis, and inevitably they do still play a notational role in this text. Perhaps what differs most is an attempt to capture ways of reasoning that remain imbued with the notion of movement.

Results

Key results include:

- A definition of *fluent* as an object within mathematics dependent on a time-like parametrisation (often denoted as $x[t]$), in Chapter 3.
- What it means for one fluent to be flowing faster than another through a time-like instant t_0, in Chapter 5.
- Properties of comparative flows, in Chapters 5 and 6.
- A characterisation of what it means for a fluent to be continuous in value through a time-like instant t_0, variously visited in Chapters 4 and 6.
- *Fluxions* and their basic properties, developed in Chapters 6 and Chapter 7. (These resemble time derivatives.)
- *Derivative of a function* in terms of fluxions (essentially a ratio), in Chapter 8.
- The product rule for fluxions and for derivatives of a function, developed in various incarnations in Chapters 6, 7 and 8.
- The chain rule for fluxions and derivatives of a function, in Chapter 8.

There are two attitudes towards fluents taken in this document:

- as interesting in their own right; and
- as a rigorous means of defining differential calculus via fluxions.

I've tried to give a sense of the former; while indicating sections which can be skipped if your primary interest is the latter.

An advantage of this being a foundational work is that much of the text may be within reach of a pre-university mathematics student or an interested scientist, with minimal prior knowledge of classical mathematical analysis.

Status

Neither the logic nor the factual statements of this document have been independently verified enough to leave me confident in its accuracy. And there may easily be better presentations of the same matter which I wasn't previously aware of.

As the author, I'd welcome any comments about accuracy or suggested improvements.

—Neil Ramsden (mail@neilramsden.co.uk, include the word "fluxions" in the Subject of any email)

Part I

Preliminaries

Chapter 1

Real Numbers

1.1 Basic Properties

Informally, the *real* numbers form a continuum. They have the ordinary arithmetic operations (+ and ×) and a linear ordering (a total order $<$). The set of real numbers is denoted $\mathbb{R}$.

Some of the concepts we work with later rely heavily on the order properties of the reals. The following lemma can sometimes be useful when working with order relations.

Lemma 1.1. *Suppose we are given two real values $a, b \in \mathbb{R}$. Then the following assertions hold.*

- *If $\forall x \in \mathbb{R} : x < a \implies x \leq b$; then $a \leq b$.*
- *If $\forall y \in \mathbb{R} : a < y \implies b \leq y$; then $b \leq a$.*

Proof. We prove the first assertion by contradiction. Suppose $\forall x \in \mathbb{R} : x < a \implies x \leq b$. Assume $b < a$ and set $c = \frac{1}{2}(a + b)$; so $b < c < a$. But $c < a$ implies $c \leq b$, which is a contradiction ※. Hence we must have $b \geq a$ as required.

The second assertion can be proved similarly, or by noting that $x < y$ is equivalent to $(-y) < (-x)$. ☐

1.2 Rational Numbers

The integer fractions of form $\frac{m}{n}$ constitute a proper subset of the set $\mathbb{R}$, called the *rational* numbers. This set is denoted $\mathbb{Q}$; while $\mathbb{Z}$ denotes the set of positive and negative integers including zero.

Every real number $x \in \mathbb{R}$ can be approximated as closely as one wishes by a suitable rational number $p \in \mathbb{Q}$. In other words, given a tolerance $\varepsilon > 0$, we can find $p = \frac{m}{n} \in \mathbb{Q}$ such that $|x - \frac{m}{n}| < \varepsilon$. This important property is equivalent to saying that the rationals form a dense subset of the reals.

1.3 Completeness

Every non-empty subset S of $\mathbb{R}$ with an upper bound in $\mathbb{R}$ has a least upper bound in $\mathbb{R}$.[12] This is a fundamental property in analysis, so let's expand on it.

We say that a real number B is an *upper bound* for the set $S \subseteq \mathbb{R}$ when, for each $x \in S$, we have $x \leq B$. The set S is *bounded above* if and only if it has an upper bound. A real number b is a *least upper bound* when it is both an upper bound and, for any other upper bound B, $b \leq B$. It is necessarily unique; moreover we're asserting that, whenever S is bounded above, the least upper bound must exist. We call this unique value the *supremum* of S and denote it $\sup S$.

Example 1.1. Suppose a is any real number and $S = \{x \in \mathbb{R} \mid x < a\}$. Then $\sup S = a$. This is also true for the set $T = \{x \in \mathbb{R} \mid x \leq a\}$; so some bounded sets will contain their supremum (but others will not).

There is a natural dual concept of greatest lower bound or *infimum* for subsets of R which are bounded below.

Remark 1.1. Completeness is a major distinguishing feature between the rationals and the reals. To see that the rationals aren't complete, consider $S = \{x \in \mathbb{Q} \mid x^2 < 2\}$. Regarding S as a subset of the real numbers, it has a supremum $\sup S = \sqrt{2} \in \mathbb{R}$; but a seminal result in mathematical history tells us that there is no supremum of S that's expressible purely as a rational number.

(If there were, it would have the form $y = \frac{m}{n}$ for integers $m, n \in \mathbb{Z}$. If $y^2 < 2$, then we could find a rational $z > y$ with $z^2 < 2$, so $z \in S$ and y wouldn't be an *upper* bound. If $y^2 > 2$, then we could find a rational $w < y$ with $w^2 > 2$, so y wouldn't be a *least* upper bound. Hence we must have $y^2 = \frac{m^2}{n^2} = 2$, which classical ancient Greek mathematicians proved was impossible.)

The characterisation of completeness here is one of several equivalent formulations. Later (in Section 2.3), we consider completeness dynamically rather than atomically, via a predicate along an interval.

1.4 The Extended Reals

(This section isn't essential—it can be skipped if your overriding interest is the way fluxions can be used to define differential calculus.)

We can extend the real numbers to include two infinite values $-\infty$ and $+\infty$. We denote this here as $\mathbb{R} \cup \{\pm\infty\}$. Ordinary arithmetic and ordering of real numbers extends to $\pm\infty$ in the obvious way (with a bit of sanitisation to avoid problematic cases, such as $\infty - \infty$).[5]

This enables us to extend 'inf' and 'sup' to mappings $\mathcal{P}\,\mathbb{R} \to \mathbb{R} \cup \{\pm\infty\}$ on the set of all subsets $\mathcal{P}\,\mathbb{R}$ of the real numbers, taking each subset to its greatest lower bound (infimum) and least upper bound (supremum) as values in $\mathbb{R} \cup \{\pm\infty\}$. When a set $S \subseteq \mathbb{R}$ is unbounded below, then we can write $\inf S = -\infty$; and if it's unbounded above then $\sup S = +\infty$.

Remark 1.2. When $S \subseteq \mathbb{R}$ is non-empty, then $\inf S \leq \sup S$, as we might expect.

When $S = \varnothing$ is the empty set and $B \in \mathbb{R}$ is any real number, then the condition $x \leq B$ is satisfied vacuously (there's no $x \in S$ contradicting it); so every $B \in \mathbb{R}$ is an upper bound, including -1, -1000, -100000, etc. We would therefore write $\sup \varnothing = -\infty$. Similarly, $\inf \varnothing = +\infty$.

We can also further extend 'inf' and 'sup' to subsets $S \subseteq \mathbb{R} \cup \{\pm\infty\}$, where S might contain none, one or both of $\pm\infty$. This then enables us to show that $\mathbb{R} \cup \{\pm\infty\}$ is complete. In fact, every $S \subseteq \mathbb{R} \cup \{\pm\infty\}$ is bounded above by some $B \in \mathbb{R} \cup \{\pm\infty\}$ and *every* $S \subseteq \mathbb{R} \cup \{\pm\infty\}$ has a supremum.

Chapter 2

Intervals

2.1 Standard results

Much of this section coincides with the Wikipedia article *Interval (mathematics)* [8].

Informally, an interval is a continuous range of real numbers. For example, the closed interval $[0, 1]$ comprises all the real numbers from 0 to 1 inclusive.

Typically, I shall denote an interval with an uppercase letter, such as H, I, J, K.

Traditionally, we can regard an interval as a set of points. In set-builder notation[14], the subset of a set S of elements $x \in S$ obeying a condition $P(x)$ may be written as $\{x \in S \mid P(x)\} \subseteq S$. For example, we can write:

$$[0, 1] = \{x \in \mathbb{R} \mid 0 \le x \le 1)\} \subseteq \mathbb{R}$$

This notation enables us to define various kinds of interval. In doing so, it's convenient to distinguish between intervals which are bounded (finite) or unbounded, and between those which are topologically open or closed (some are neither).

The main intervals of interest in this document are subsets of the real numbers $\mathbb{R}$. In the following definitions, a and b are real numbers with $a \le b$. The direction of the square brackets '[' or ']' is used to indicate inclusion or exclusion of endpoints (and hence whether the interval is open or closed).

Definition 2.1. A non-empty *bounded*, or *finite*, *open interval* has the form:

- $]a, b[\ = \{x \in \mathbb{R} \mid a < x < b\}$;

Its *length* is then $b - a > 0$.

Definition 2.2. A non-empty *bounded*, or *finite*, *closed interval* has the form:

- $[a, b] = \{x \in \mathbb{R} \mid a \leq x \leq b\}$;

Its *length* is again $b - a$.

Remark 2.1. The empty set $\varnothing$ is topologically both open and closed. Because we stipulate that $a \leq b$, every interval of the form $[a, b]$ is necessarily non-empty. When $a = b$, its interior is the empty set $]a, a[= \varnothing$.

Whenever it might be convenient, in this document I'll regard the empty set as an empty interval with undefined endpoints and length.

Definition 2.3. An *unbounded open interval* has one of three forms:

- $]-\infty, b[= \{x \in \mathbb{R} \mid x < b\}$;
- $]a, +\infty[= \{x \in \mathbb{R} \mid a < x\}$;
- $]-\infty, +\infty[\; \equiv \mathbb{R}$;

Its length is infinite.

Definition 2.4. An *unbounded closed interval* has one of three forms:

- $]-\infty, b] = \{x \in \mathbb{R} \mid x \leq b\}$;
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- $]-\infty, +\infty[\; \equiv \mathbb{R}$;

Its length is infinite.

Notice that $\mathbb{R} \equiv \,]-\infty, +\infty[$ is both open and closed.

Definition 2.5. The *interior* of an interval I is the largest open interval that is contained in I; it therefore comprises the set of points in I which are not endpoints of I.

I'll denote it I^o.

Definition 2.6. The *closure* of I is the smallest closed interval that contains I. When $I = \,]a, b[$ is non-empty and finite (so $-\infty < a < b < +\infty$), its closure comprises the set I augmented with its two finite endpoints a and b.

Definition 2.7. An interval I is a *subinterval* of interval K if I is a subset of K. I'll simply write this as $I \subseteq K$.

2.2 Bespoke Results

This section introduces terminology and results specifically geared to the needs of fluents and their fluxions. We will often be interested in the time domain, so typically use t_0, t_1 and t_2 in $\mathbb{R}$ to denote these time-like values.

2.2.1 Overlapping Intervals

In this document, I define two intervals as overlapping if they have at least one point in common; or equivalently they have non-empty intersection.

Theorem 2.1. *If two finite closed intervals $I = [a, b]$ and $J = [c, d]$ have non-empty intersection $I \cap J \neq \varnothing$, then:*

- *their intersection is the closed interval $I \cap J = [e, f]$, where:*

$$\begin{aligned} e &= \max\{a, c\}, \\ f &= \min\{b, d\}; \end{aligned}$$

- *and their union is the closed interval $I \cup J = [e', f']$, where:*

$$\begin{aligned} e' &= \min\{a, c\}, \\ f' &= \max\{b, d\}. \end{aligned}$$

Proof. Deal with the intersection $I \cap J$ first, putting $H = [e, f]$. We wish to prove that $I \cap J = H$

We note that if $I \cap J$ is non-empty, then for some t, we have $t \in I \cap J$; so $a \leq t \leq b$ and $c \leq t \leq d$. Therefore $\max\{a, c\} \leq t \leq \min\{b, d\}$ and H is a non-empty well-defined finite closed interval containing t.

Suppose $s \in I \cap J$. Then $a \leq s \leq b$ and $c \leq s \leq d$. So $\max\{a, c\} \leq s \leq \min\{b, d\}$; and $s \in H$. This is true for all $s \in I \cap J$; therefore $I \cap J \subseteq H$.

Now suppose $s \in H$. Then $\max\{a, c\} \leq s \leq \min\{b, d\}$. So $a \leq s \leq b$ and $c \leq s \leq d$; and $s \in I \cap J$. This is true for all $s \in H$; therefore $H \subseteq I \cap J$.

Combining these two results, $H = I \cap J$, as claimed.

The proof that $I \cup J = [e', f']$ is somewhat analogous, putting $K = [e', f']$. We wish to prove that $I \cup J = K$.

Note that if $I \cap J$ is non-empty, then a similar argument to the one above confirms that $\min\{a, c\} \leq t \leq \max\{b, d\}$ and K is a non-empty well-defined finite closed interval containing t.

Suppose $s \in I \cup J$, so $s \in I$ or $s \in J$ (or both); ie. $a \leq s \leq b$ or $c \leq s \leq d$. Then $\min\{a, c\} \leq s \leq \max\{b, d\}$; and $s \in K$. This is true for all $s \in I \cup J$; therefore $I \cup J \subseteq K$.

Now suppose $s \in K$. Then $\min\{a, c\} \leq s \leq \max\{b, d\}$. So $a \leq s$ or $c \leq s$; and $s \leq b$ or $s \leq d$.

Suppose without loss of generality that $a \leq c$; then necessarily $a \leq s$. If $s \leq b$, then $a \leq s \leq b$ and $s \in I$. If $\neg(s \leq b)$ (ie. $b < s$), then $s \leq d$. But then $c \leq b < s$, giving $c < s \leq d$ and $s \in J$. We've shown that for all $s \in K$ either $s \in I$ or $s \in J$; therefore $K \subseteq I \cup J$.

Combining these two results, $K = I \cup J$, as claimed. □

2.2.2 Non-trivial Intervals Straddling a Given Point

A fluent has a time-like parameter taking values in a given interval, and we will often think of the parameter as traversing that interval from the start to the finish. (The word 'parameter' here has the same sense as for a parametric curve, rather than a statistical parameter.) The value of the fluent itself can then be thought of as co-varying in some abstract way as its parameter changes. We will often be interested in the way a fluent changes as this notional time progresses through an instant t_0.

Unlike modern analysis which more often deals with open intervals, it seems more natural to develop fluents and fluxions by working with closed intervals, which include their start and finish points.

These considerations prompt the following two definitions and their simple corollary. They apply to any interval, but are especially geared towards closed intervals.

Definition 2.8. An interval I *straddles* or *strictly contains* a value t_0 iff $t_0 \in I^o$.

Example 2.1. The closed interval $[3, 4]$ straddles π.

Example 2.2. More generally, the closed interval $I = [t_1, t_2]$ straddles t_0 iff $t_1 < t_0 < t_2$

Definition 2.9. An interval I is *non-trivial* iff I^o is non-empty.

Corollary 2.2. *If an interval I straddles a value t_0, then it must be non-trivial.*

Sometimes it's necessary to find a closed sub-interval $I \subseteq D$ straddling $t_0 \in D$, where D is a finite open interval. The next lemma provides one solution. (In fact, there are many possible solutions.)

Lemma 2.3. *Suppose $D =]T_1, T_2[$ is a finite open interval containing a value $t_0 \in D$.*

Then there is a closed sub-interval $I = [t_1, t_2] \subseteq D$ straddling t_0 for some $t_1, t_2 \in D$.

Proof. Set $t_1 = \frac{1}{2}(T_1 + t_0)$; and set $t_2 = \frac{1}{2}(t_0 + T_2)$.

It's easy to check that $I = [t_1, t_2]$ has the desired properties, noting that $T_1 < t_0 < T_2$. □

2.2.3 Upper and Lower Bounds

Given a non-empty interval I, we can denote its endpoints as $\inf I$ and $\sup I$, using the extended infimum and supremum functions defined in Section 1.4. Alternatively, when I is a finite closed interval, we can use the less intimidating notation $\min I$ and $\max I$ (minimum and maximum points of I).

Definition 2.10. We say a closed interval $I = [t_1, t_2]$ or $]-\infty, t_2]$ is *strictly bounded above* by $b \in \mathbb{R}$ iff $t_2 < b$ (equivalently $\sup I < b$).

I'll write this as $I < b$ or (more rarely) $b > I$.

Definition 2.11. Similarly, a closed interval $I = [t_1, t_2]$ or $[t_1, +\infty[$ is *strictly bounded below* by $a \in \mathbb{R}$ iff $a < t_1$ (equivalently $a < \inf I$).

I'll write this as $I > a$ or $a < I$.

The second definition is the obvious dual relation. There are two obvious corollaries (dual to each other) on how this property interacts with the ordering on real numbers.

Corollary 2.4. *Suppose that real numbers b and d satisfy $b < d$. Suppose also that the closed interval I is strictly bounded above by b, so $I < b$. Then I must also be strictly bounded above by d.*

In other words, if $I < b$ and $b < d$, then $I < d$.

Corollary 2.5. *Suppose that real numbers a and c satisfy $c < a$. Suppose also that the closed interval I is strictly bounded below by a, so $a < I$. Then I must also be strictly bounded below by c.*

In other words, if $c < a$ and $a < I$, then $c < I$.

We can apply these corollaries in a natural way to express more complex relationships such as $c < a < I < b < d$.

2.3 Completeness along an Interval

The classical set-based characterisation of the completeness of the real numbers in Section 1.3 is well-established and succinct; but it can seem distant from our intuitions in trying to capture the nature of a continuum, such as a real interval $I = [a, b] \subseteq \mathbb{R}$. An alternative approach is to consider a $\{\text{true}, \text{false}\}$ property (a *predicate*) that applies along the interval I, changing from true to false along the way.

When P is a predicate, I'll typically write $\neg P$ for the negative of P (read "not P"); and $P(x)$ when the property P is true of some mathematical object x. Hence, I write $\neg P(x)$ when x doesn't have the property P, ie. when the property denoted by P is false for x.

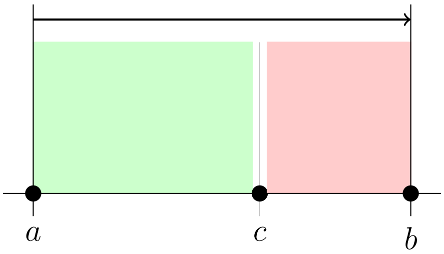

Figure 2.1: A bounded lower predicate P on an interval $[a, b]$ can be represented by a green region where P holds and red for $\neg P$. It is true at a, becoming false by point b. A lower predicate ensures that there is at most one transition from true to false. Intuitively, as one travels from a to b there will be a unique changeover point c, where $P(x)$ is true for $x < c$ and false for $x > c$.

Imagine a predicate P that applies to any point along a finite closed non-trivial interval $I = [a, b]$, and which starts true at a and ends up false at b. Intuitively, as one traverses the interval, there must be at least one changeover, where the value of P goes from true to false. Limiting ourselves to the case where there's just one such changeover motivates the following definitions, which borrow loosely from order theory[17].

Definition 2.12. A *lower property* of the real numbers on an interval $I \subseteq \mathbb{R}$ is a unary predicate $P(x)$ defined for $x \in I$ for which:

- for any $x, y \in I$ with $x < y$, $P(y) \Rightarrow P(x)$.

Dually, Q is an *upper property* on I iff $\neg Q$ is a lower property on I. In other words:

- for any $x, y \in I$ with $x < y$, $Q(x) \Rightarrow Q(y)$.

Notice that a lower property must have at most one changeover point (as does an upper property).

Definition 2.13. When I is a finite closed interval $I = [a, b]$, we say a lower property P is a *bounded lower property* on I if additionally:

- $P(a)$; and

- $\neg P(b)$.

The interval I is then necessarily non-trivial (a and b must be distinct points). Dually, Q is a *bounded upper property* on I iff $\neg Q$ is a bounded lower property on I.

A bounded lower property has exactly one changeover point, as does a bounded upper property (in each case, there's at least one and no more than one). The following theorem takes a classicist standpoint. It shows that the set-based characterisation of completeness in Section 1.3 implies that the changeover point exists as a real number.

Theorem 2.6. *We assume that every non-empty subset of* $\mathbb{R}$ *with an upper bound has a supremum in* $\mathbb{R}$*. Now suppose that the finite closed non-trivial interval* $I = [a, b]$ *has a bounded lower property* P*. Then there is a unique point* $c \in I$ *where* $\forall x \in I$*:*

- $x < c \Rightarrow P(x)$*; and*
- $x > c \Rightarrow \neg P(x)$*;*

  or, equivalently: $P(x) \Rightarrow x \leq c$.

Proof. To prove existence, put $S = \{x \in I \mid P(x)\}$. Then S is non-empty (because $a \in S$) and bounded by b, so has a supremum $c \leq b$. Also, $a \in S$ implies $a \leq c$, so $c \in I$. The concluding implications $x < c \Rightarrow P(x)$ and $x > c \Rightarrow \neg P(x)$ then follow from $c = \sup S$.

We could prove uniqueness by showing that any candidate c' must be the supremum of S and hence, by uniqueness of the supremum, be equal to c. Instead, we assume there are two unequal candidates satisfying the bulleted conditions and show by first principles that this leads to a contradiction.

Label such candidates c and c' with $c < c'$. Now put $c'' = \frac{1}{2}(c + c')$; so that $c < c''$, which implies $\neg P(c'')$; and $c'' < c'$, implying $P(c'')$; which is a contradiction ※. This confirms that if c is a candidate, then it must be unique. $\square$

Remark 2.2. It may be that c equals one of the endpoints a or b. In that case, one of the resulting bulleted implications in the theorem will be true vacuously.

Example 2.3. The conclusion of Theorem 2.6 fails for rational intervals. For example, it is well-established that the square root of 2 is irrational[15]. Define a predicate $P(x) \equiv x^2 < 2$ on the rational interval $I = [1, 2] \subseteq \mathbb{Q}$. Then $P(1)$ holds but not $P(2)$, so P is a bounded lower property on I. We show by contradiction that there is no point $c \in I$ with the stated properties.

Assume such a c exists. It is necessarily rational, so cannot equal $\sqrt{2}$ and must be either less than or greater than $\sqrt{2}$; hence its square must be either less than or greater than 2.

Assume first that $c^2 < 2$. Define a new point:

$$\begin{aligned} d &= 1 + \cfrac{1}{2 + \cfrac{1}{1+c}} \\ &= \frac{4+3c}{3+2c}. \end{aligned}$$

Since c is rational, so is d. We now show that $c < d$ and $d^2 < 2$:

$$\begin{aligned} d - c &= \frac{(4+3c) - (3+2c)c}{3+2c} \\ &= 2\,\frac{2-c^2}{3+2c} \\ &> 0 \end{aligned}$$

and:

$$\begin{aligned} 2 - d^2 &= \frac{2(3+2c)^2 - (4+3c)^2}{(3+2c)^2} \\ &= \frac{2-c^2}{(3+2c)^2} \\ &> 0. \end{aligned}$$

But $d > c \Rightarrow \neg P(d)$, while we have shown that $P(d)$ holds, which is a contradiction ※.

A similar contradiction occurs when we assume $c^2 > 2$, since then we can show $d < c$ and $\neg P(d)$.

Hence there is no such $c \in I \subseteq \mathbb{Q}$ when we restrict ourselves to the rational numbers.

Notice as an aside that if we limit ourselves to the counting numbers $\mathbb{N}$, then $I \subseteq \mathbb{N}$, and there is at least one such c (ie. 1 or 2), but it is not unique.

Definition 2.14. For P and c in Theorem 2.6, we call c the *supremum of property* P, and write '$c = \sup P$'.

The *infimum* of an upper property Q is defined dually in the obvious way and is denoted by '$\inf Q$'.

This suggests a different characterisation of completeness: that every bounded lower property on a finite closed interval has a supremum. The next theorem confirms that this is strong enough to prove the existence of the supremum of any a non-empty subset of $\mathbb{R}$ with an upper bound. It's therefore equivalent to other classical formulations of completeness as well, such as Cauchy completeness or Dedekind completeness.

Given a set $S \subseteq \mathbb{R}$, an obvious corresponding property is $P(x)$ iff $x \in S$; but there's no guarantee that this will be a lower property on a suitable interval. There is, however, a simple construction that provides this guarantee.

Definition 2.15. If $I \subseteq \mathbb{R}$ is an interval and $P(x)$ is a predicate defined for $x \in I$, then the *lower closure* of P is a predicate denoted $P^{\downarrow}$ or $P^{\downarrow I}$ where $P^{\downarrow I}(x)$ iff for some $x^* \in I$ with $x^* \geq x$, we have $P(x^*)$.

The *upper closure* is the obvious dual, denoted $P^{\uparrow}$ or $P^{\uparrow I}$.

Lemma 2.7. *If $I \subseteq \mathbb{R}$ is an interval and $P(x)$ is a predicate defined for $x \in I$, then the lower closure of P defines a lower property $P^{\downarrow}$ on I.*

Proof. The proof follows straight-forwardly from the definitions. ☐

Theorem 2.8. *Assume that every bounded lower property on a finite closed interval has a supremum. Then every non-empty subset of the reals with an upper bound has a supremum which is a real number.*

Proof. We proceed in several steps.

- *Given the set S, to construct a property P on an interval I*: Take a non-empty set $S \subseteq \mathbb{R}$ which is bounded above by $b \in \mathbb{R}$. If $b \in S$, then $b = \sup S$ and we're done; so assume $b \notin S$.

  There is at least one element $a \in S$. We must then have $a < b$; and hence the interval $I = [a, b] \subseteq \mathbb{R}$ is non-trivial. Define a property P with $P(x)$ iff $x \in S \cap I$, so $P(a)$ and $\neg P(b)$.

- *To construct the closure $P^{\downarrow I}$ and its supremum c*: Now consider the downward closure $P^{\downarrow I}$ on I, which by construction is a lower property. Again $P^{\downarrow I}(a)$ and $\neg P^{\downarrow I}(b)$, so $P^{\downarrow I}$ is bounded and has a supremum $c = \sup P^{\downarrow I} \in I$.

- *To show c is an upper bound for S*: Now take $x \in S$, so $x \leq b$. If $x > c$, then $x \in I$ and, by definition of c, $\neg P^{\downarrow I}(x)$. But $x \in I$ also means $x \in S \cap I$, ie. $P(x)$, and hence $P^{\downarrow I}(x)$; which is a contradiction ※. Hence $x \leq c$ and c is an upper bound for S.

- *To show $c = \sup S$*: Now suppose that $d < c$ is another upper bound for S; which is necessarily in I since $a \leq d$. Put $e = \frac{1}{2}(c + d) \in I$. This would imply $P^{\downarrow I}(e)$, because $e < c = \sup P^{\downarrow I}$. But this requires there to be $e^* \in I$ with $e \leq e^*$ and $P(e^*)$, ie. $e^* \in S$. Since $d < e$, we then would have $d < e^*$, which contradicts d being an upper bound for S ※. Hence we must have $c \leq d$, ie. c is the lowest upper bound of S.

□

This justifies the assertion that completeness of the real numbers can be expressed by stating that every bounded lower property on a finite closed interval has a supremum. Its proof suggests a more general predicate-based formulation of completeness along an interval.

In expressing this more general result, it can be helpful to think of a unary predicate $P(x)$ for $x \in I$ as a function $P\colon I \to \{\text{true}, \text{false}\}$, where the domain I is an implicit component of P.

Theorem 2.9. *Suppose that I is a finite closed non-trivial real interval $I \subseteq \mathbb{R}$, and $P\colon I \to \{\text{true}, \text{false}\}$ is a unary predicate (so $P(x)$ is defined for each $x \in I$).*

For the purposes of this theorem, we define $d \in I$ to be an upper bound of predicate P *iff for each $x \in I$, $P(x) \Rightarrow x \leq d$.*

Then there is a unique point $c \in I$, where for each upper bound $d \in I$ and each $x \in I$:

- *$c \leq d$; and*
- *c is an upper bound of predicate P;*

  ie: $P(x) \Rightarrow x \leq c$;

  or, equivalently: $x > c \Rightarrow \neg P(x)$.

Proof. We can prove this from the classical set-theoretic formulation of completeness or from completeness along an interval.

- *Assuming that every non-empty subset of $\mathbb{R}$ with an upper bound has a supremum in $\mathbb{R}$*: Define $S = \{x \in I \mid P(x)\}$. Put $I = [a, b]$, where $a, b \in \mathbb{R}$. If S is empty, put $c = a$; otherwise put $c = \sup S \leq b$.

- *Assuming completeness along an interval*: We can adapt the proof of Theorem 2.8.

Either approach affirms the result. □

Remark 2.3. Note that when the set S in the proof is empty, the definition of supremum of P isn't the same as $\sup S$ when S is thought of as a subset of $\mathbb{R}$.

Definition 2.16. When $P\colon I \to \{\text{true}, \text{false}\}$ is a unary predicate on a finite closed non-trivial real interval $I \subseteq \mathbb{R}$, we call c of Theorem 2.9 the *supremum of* P and write $c = \sup P$.

We define the *infimum* of $P\colon I \to \{\text{true}, \text{false}\}$ dually, denoted $\inf P$.

Corollary 2.10. *Suppose that* I *is a finite closed non-trivial real interval* $I \subseteq \mathbb{R}$*; and* $P\colon I \to \{\text{true}, \text{false}\}$ *is a lower unary predicate (ie. for all* $x, y \in I$*, if* $P(y)$ *holds and* $x < y$*, then* $P(x)$ *holds).*

Then there is a unique point $c \in I$ *where* $\forall x \in I$*:*

- $x < c \Rightarrow P(x)$*; and*
- $x > c \Rightarrow \neg P(x)$*;*

  or, equivalently: $P(x) \Rightarrow x \leq c$*.*

Proof. This follows fairly directly from Theorem 2.9, noting that $P(x)$ being false for a lower property P implies that x is an upper bound of P and hence $x \geq c = \sup P$. $\square$

We'll generally appeal to completeness along an interval in future proofs, rather than the classical set-theoretic form.

Part II

Fluents and Flow Velocity

Chapter 3

Fluents

3.1 Introducing Fluents

3.1.1 Preamble

In a nutshell, we can think of a fluent as a variable whose value is in a state of continuous progression. To get started, let's expand on this informally.

Suppose that the variable x is the length of the side of a square and y its area. We could graph this relationship by drawing the curve $y = x^2$, where we think of y as the dependent variable and x as the independent one.

Rather than thinking of x and y as representing one of a number of possible values (static points), we will generally regard them here as dynamic variables, which flow continuously through notional time. To emphasise this view, I borrow the name *fluent* for them and write $x[t]$ and $y[t]$ to suggest a time-based parametrisation. The word 'time' here is to be taken figuratively in priming our intuition rather than literally—when we imagine a continuously growing square there is no physical clock in our imagination counting off the seconds.

Now, in physics we'll often be interested in variables in a continuous process of change. Imagine for example, if the base of our square is growing continuously in size. We can picture this state of affairs as a point in continuous motion along the x-axis reflecting the growing length of side. There will then be a corresponding point that moves along the curve; and a point representing the square's area travelling continuously along the y-axis.

When envisaging what I've called a 'continuous' flow, what I really have in mind is that the time-like parameter t is continuous. Often it will be continuously advancing from a value t_1 to a larger value t_2. In more general cases, it's possible that a fluent $x[t]$ or $y[t]$ may involve a sudden jump or go back on itself or always stay at the same value. Later on, we'll explore what it means for the fluent itself to be continuous.

3.1.2 Definition of a Fluent

Definition 3.1. A *fluent* of *type* X, *value set* $U \subseteq X$, *indexing interval* $I \subseteq \mathbb{R}$ and *evaluation function* $\nu\colon I \to U$ is a tuple (X, U, I, ν); where X is some suitable space, $U \subseteq X$ is a set of possible values of the fluent, I is a finite non-trivial closed interval in $\mathbb{R}$, and ν maps a parametric value $t \in I$ to a corresponding value $\nu(t) \in U$.

This document mainly deals with fluents taking real values, so that $X = \mathbb{R}$. In more general settings, it may for example be that X is a Banach space[4] (such as ordinary 2- or 3-dimensional Euclidean space).

The value set U is simply the codomain of the evaluation function within the possibly larger space X. In general, there is no requirement for ν to be surjective, ie. for the fluent to take all values in U; nor any reason to prohibit $U = X$. For example, if the fluent represented the area of a square, we might expect U to be the set of non-negative real numbers.

In some contexts, it may be more convenient to require U to be the image of ν, in which case the function $\nu\colon I \to U$ will be surjective (but not necessarily injective). In that case, we can call U the *image set* of x. Such an image set must be non-empty; but it may be as small as a single element, when x is a constant fluent.

Unless specified otherwise, in most of what follows we take $X = \mathbb{R}$ and assume that $U \subseteq \mathbb{R}$ is 'large enough' to contain the values of interest.

3.1.3 Fluent Continuations

The stipulation that I must be both finite and closed is convenient, but can seem unnecessarily restrictive. In most contexts it's useful to be able to evaluate the fluent at the start and end of I, but some evaluation functions naturally lend themselves to being defined on more general subsets of $\mathbb{R}$. For example, one might wish to define $\nu(t) = \frac{1}{t}$ for all $t > 0$ or (more thoroughly) for all $t \neq 0$.

We could accommodate such cases by regarding them as a *continuation* of an ordinary fluent, defined on a more general non-trivial domain $D \subseteq \mathbb{R}$. We can then restrict ourselves to a suitable finite closed indexing interval $I \subseteq D$ for our working fluent according to context. Because of this, when we describe an interval I as an indexing interval in this document, we imply that it is non-trivial, finite and closed, so takes the form $[T_1, T_2]$ with $T_1, T_2 \in \mathbb{R}$ and $T_1 < T_2$.

Some, but not all, of the following notation and constructions for fluents apply in an obvious way to fluent continuations, eg. by replacing a fluent's indexing interval I by a fluent continuation's domain D.

3.1.4 Notation

When there is no ambiguity, we will usually denote a fluent $x = (X, U, I, \nu)$ simply as x. We then write $x[t]$ for the value $\nu(t)$ of x at the parametric value t; or equivalently $t \mapsto x[t]$ in place of $t \mapsto \nu(t)$.

In this example, the choice of letter t emphasises the view of a time-like parametrisation. If $I = [T_1, T_2]$, then we typically think of t as traversing I from T_1 to T_2, which gives the fluent its dynamic quality. The use of square brackets in $x[t]$ reminds us that the parameter t denotes a *notional time*, analogous to physical time, but not necessarily time itself.

In keeping with this close analogy with physical time, we'll often refer to a specific parametric value as an 'instant', especially when we discuss flow velocity in subsequent chapters.

3.1.5 Differing Parametrisations

Informally, we might regard a fluent as representing a dynamic variable that takes on values in $U \subseteq X$. By 'dynamic variable' I mean a more primitive form of mathematical object, which we can think of as varying in value within U in some unspecified way. We may then wish to assign different evaluations $\nu\colon I \to U$ and $\mu\colon H \to U$ to the same dynamic variable.

Here, μ and ν are two different functions with two (possibly) different domains H and I, both of them closed intervals. A convenient, if sometimes ambiguous, shorthand for this situation is then to use two different symbols for the parameter, eg. notationally define $x[t] \equiv \nu(t)$ in contrast to $x[s] \equiv \mu(s)$. The important thing in this example is that we choose a symbol other than t when we write $x[s]$ to signal the new parametrisation μ. The notation is akin to a change of variable in a physical equation, eg. expressing the height of a projectile as a function of time versus a function of distance travelled.

We will typically use this convention when subsequently we talk of different parametrisations, even though what more formally we have in mind is different evaluation functions. In other words, when the parametrisation (evaluation function) is understood, we will often refer to the fluent $x = (X, U, I, \nu)$ simply as x. If we need to be more specific, we may write this as $x[t]$; and conveniently distinguish parametrisations as eg. $x[s]$ and $x[t]$, where the mappings of s and t to values of x may differ.

Example 3.1. Suppose we wish to express the y-coordinate of the parabola $y = x^2$ as a fluent $y[s] = (\mathbb{R}, U, H, \mu(s))$. We might give it an evaluation function $\mu(s) = s^2$ with the interval $H = [-2, +2]$ as the domain of s. We have signalled this choice of parametrisation as $y[s]$. We could associate this fluent with traversing the curve with constant horizontal velocity from left to right.

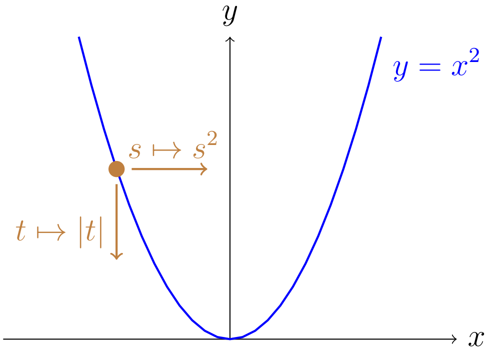

Figure 3.1: Graph of $y = x^2$. The fluents $y[s] = s^2$ and $y[t] = |t|$ have different parametrisations, which can be thought of respectively as traversing the curve at constant speed with a horizontal or a vertical motion.

We might also consider a point traversing the parabola from left to right, downwards then upwards with a constant vertical speed (except possibly at $y = 0$). One solution might be a new parametrisation denoted by $y[t] = (\mathbb{R}, U, I, \nu(t))$, where now $y[t] = \nu(t) = |t|$ and the domain of t is $I = [-4, +4]$.

We are able to distinguish the parametrisations as $y[s]$ and $y[t]$. Both parametrisations have the same image $U = [0, 4]$ (decreasing to zero then increasing) and they have the same geometric source. We might therefore think of them as parametrising the same variable; but they are distinct fluents with various differing properties.

In some contexts there may be too many ambiguities lurking behind this notation. Suppose we have a variable $x \in U \subseteq X$. To avoid any confusion, we may wish to distinguish two fluents taking the same values as x by decorating the symbol x as well as the symbols in the parametrisation. For example, we might denote the fluents as x^{A} and x^{B}, so that we can unambiguously distinguish $x^{\mathsf{A}} = (X, U, I^{\mathsf{A}}, \nu^{\mathsf{A}})$ from $x^{\mathsf{B}} = (X, U, I^{\mathsf{B}}, \nu^{\mathsf{B}})$. We'll employ this notation in Chapter 8 when we show that the fluent-based foundation for the derivative is well-defined.

3.2 Basic Constructions

3.2.1 Relabelling a Parametrisation

An important kind of reparametrisation of a fluent $x = (X, U, I, \nu)$ occurs when we have a function $r \colon H \to I$, which can then provide a *relabelling* of the values of x by an indexing interval $H \subseteq \mathbb{R}$. Whenever $t = r(s)$ for $t \in I$ and $s \in H$, then we can define $x[s] = x[t]$. Depending on the nature of r and H, the image of $r \colon H \to I$ may be the whole of I or a proper subset.

Extending the notation slightly, we can think of $x[t] = (X, U, I, \nu(t))$ and $x[s] = (X, U, H, \mu(s))$ as the same dynamic variable expressed as two closely related fluents with differing evaluation functions ν and μ, where $\nu(r(s)) = \mu(s)$ for each $s \in H$. This relation can be expressed as the composition of functions $\nu \circ r = \mu$. It suggests using the notation $x \circ r$ as a shorthand for the fluent (X, U, H, μ).

Example 3.2. A trivial example of a relabelling is when $H \subseteq I$ and r is the inclusion function $H \to I$ with $r(s) = s$ for each $s \in H$. This can be useful when I is in some sense 'too large', eg. it needs to be restricted for a required condition to be met. We can then substitute for $\mu = \nu|_H$ and think of (X, U, I, ν) and $(X, U, H, \nu|_H)$ as essentially the same fluent within H.

This example leads naturally to the following definition.

Definition 3.2. Given a fluent $x = (X, U, I, \nu)$ and a non-trivial closed interval H with $H \subseteq I$, we call the resulting fluent $(X, U, H, \nu|_H)$ the *restriction of x to H* and denote it by $x|_H$.

Note that if instead x is a fluent continuation and I is a finite non-trivial closed interval within the domain of x, then we can fairly naturally extend this notation to construct the fluent $x|_I$, which is the restriction of x to the indexing interval I.

Very often in the analysis of fluents we're interested in what's happening in a suitable neighbourhood of a given parameter value t_0, rather than the whole of the indexing interval. Because of this, we will often choose to restrict a fluent to a suitable sub-interval straddling t_0.

The indexing interval and evaluation function determine a direction for traversing the values of a fluent x. Another simple kind of relabelling constructs a dual fluent, which traverses these values in the opposite direction.

Definition 3.3. The *opposite of a fluent* $x = (X, U, I, \nu)$ is the fluent $\operatorname{opp} x = (X, U, H, \mu)$ (also denoted x^{op}), where the closed interval $H = [-\max I, -\min I]$ and $\forall s \in H\,(\mu(s) = \nu(-s))$.

By analogy, we call H the *opposite* of interval I or the *negative* of I, which we might denote as $\operatorname{opp} I$, I^{op} or $-I$.

Remark 3.1. Notice that the operator opp is idempotent, ie. $\operatorname{opp}\operatorname{opp} x = x$ and $\operatorname{opp}\operatorname{opp} I = I$.

As suggested by the name, an increasing fluent flow becomes a decreasing one in the fluent's opposite. In general, the opposite of a fluent leads to a duality in inequalities concerned with how a fluent progresses (and a potential duality in sign). For example, in comparing how two fluents change as they traverse an indexing interval, inequalities will be reversed when the corresponding increments in their opposites are compared.

Because relabelling a fluent reduces to a function composition with its evaluation function, relabelling has an associative property. As a result, if $p\colon H \to I$ and $q\colon G \to H$ are two successive relabellings of a fluent $x = (X, U, I, \nu)$, we can unambiguously write $x \circ p \circ q$, identifying the resulting fluent with both $(x \circ p) \circ q$ and $x \circ (p \circ q)$.

We can exploit this observation in deducing the following lemma.

Lemma 3.1. *Given a fluent $x = (X, U, I, \nu)$ and relabellings $r\colon H \to I$, $p\colon H \to I$ and $q\colon G \to H$; then we can define a relabelling $r^{\mathrm{op}}\colon H^{\mathrm{op}} \to I^{\mathrm{op}}$ of the fluent x^{op} with the following properties.*

- *For all $(-s) \in H^{\mathrm{op}}$, $r^{\mathrm{op}}(-s) = -r(s) \in I^{\mathrm{op}}$.*
- $\operatorname{opp}\operatorname{opp} r = r$.
- $(x \circ r)^{\mathrm{op}} = x^{\mathrm{op}} \circ r^{\mathrm{op}}$.
- $(p \circ q)^{\mathrm{op}} = p^{\mathrm{op}} \circ q^{\mathrm{op}}$.

We've used the obvious notation $\operatorname{opp} r \equiv r^{\mathrm{op}}$.

Proof. The first bulleted statement defines r^{op}. The rest follow from this by expanding the identities involved.

Alternatively, define the negation function $n\colon \mathbb{R} \to \mathbb{R}$, where $n(s) = -s$. Then $n|_{I^{\mathrm{op}}}\colon I^{\mathrm{op}} \to I$ induces a relabelling of x, and $n|_I$ a relabelling of x^{op}, and so on. The results follow from associativity of composition of relabellings, after noting:

- $n|_I \circ n|_{I^{\mathrm{op}}} = \mathrm{id}|_{I^{\mathrm{op}}}$;
- $n|_{I^{\mathrm{op}}} \circ n|_I = \mathrm{id}|_I$;
- $\operatorname{opp} x = x \circ n|_{I^{\mathrm{op}}}$; and
- $\operatorname{opp} r = n|_{I^{\mathrm{op}}} \circ r \circ n|_I$;

where we've denoted the identity function on $\mathbb{R}$ by $\mathrm{id} = n \circ n$. □

3.2.2 Functions of a Fluent

Suppose that we have a fluent $x = (X, U_x, I, \nu_x)$. If we also have a function $f\colon U_x \to Y$, then we can define a new fluent $y = (Y, U_y, I, \nu_y)$ for suitable $U_y \subseteq Y$. We define $y[t] = f\,(x[t])$ or more formally $\nu_y = f \circ \nu_x$ as a composition of functions. We might write this $y = f \circ x$ by analogy or simply $y = f(x)$.

Once we have the concept of the *fluxion*, this construction is important in defining the *derivative of a function* in terms of the fluxions induced by the two fluents x and y.

Remark 3.2. Both the relabelling of a fluent (shown above as $\nu \circ r = \mu$) and a function of a fluent (given here by $\nu_y = f \circ \nu_x$) involve composition of functions. The difference between them can be seen by contrasting the opposite $y_{\text{opp}}[s] = x[-s]$ of fluent x with its negation $y_{\text{neg}}[t] = -x[t]$.

3.2.3 Simple Operations and Relations

Take two fluents $x = (X, U_x, I, \nu_x)$ and $y = (X, U_y, I, \nu_y)$ with the same type X and indexing interval I. Where an operation such as addition makes sense on the elements of X, then that operation can be ported to x and y as fluents, eg. $x + y$, etc, to produce a new fluent. The same is true of a relation such as 'less than'.

For example, when $X = \mathbb{R}$ (or any other additive space) we can define the sum $x + y$ to be the fluent $(X, U_{x+y}, I, \nu_{x+y})$, where for each $t \in I$:

$$\nu_{x+y}(t) = \nu_x(t) + \nu_y(t),$$

and the set U_{x+y} is some suitable subset of $\mathbb{R}$ or X. We can conveniently write the value at t to be $(x + y)[t] = x[t] + y[t]$.

Similarly, we can derive fluents such as $2x$ where $(2x)[t] = 2\,x[t]$, which is a special case of a function of the fluent x. As we'd expect, $2x = x + x$ as fluents. When $X = \mathbb{R}$ then the product $x.y$ or plain xy of fluents x and y makes sense, with the obvious definition of $(x.y)[t] = x[t]\,y[t]$.

Remark 3.3. The definitions of sums and products still hold when x and y have differing indexing intervals I and K, provided $I \cap K$ is non-trivial and we restrict x and y to the common interval $H = I \cap K$.

Again, when $X = \mathbb{R}$ (or any other ordered set), then the expression $x < y$ makes sense and is equivalent to $\nu_x < \nu_x$ as functions. In other words, $x < y$ iff for all $t \in I$, $x[t] < y[t]$. These then have the usual ordering properties as when ordering functions. When $a \in X$, we can also write $x > a$ with the obvious meaning. For example, when x is a real-valued fluent with $x > 0$, we can define a new fluent for the reciprocal $\frac{1}{x} > 0$, where for each $t \in I$:

$$\left(\frac{1}{x}\right)[t] = \frac{1}{x[t]} > 0.$$

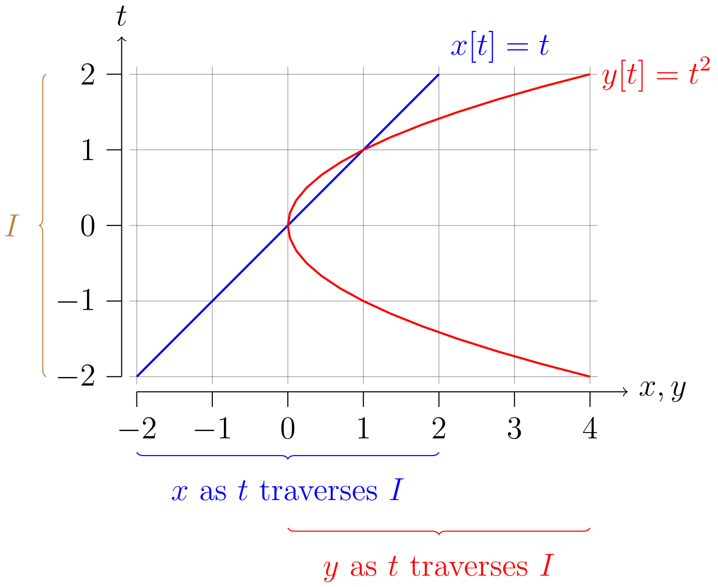

Figure 3.2: The fluents $x[t] = t$ and $y = x^2$ with indexing interval $I = [-2, +2]$.

3.3 Depicting a Fluent

A general fluent $x = (X, U, I, \nu)$ always has a finite indexing interval $I \subseteq \mathbb{R}$. One way to represent a fluent diagrammatically is to consider the time-like parameter t as traversing I vertically, with parallel copies of the space X stacked one on top of another at a height determined by t. We could denote this copy as X_t. One can then think of the evaluation function as picking out a value $\nu(t) \in X_t$ from the corresponding copy of X.

The most amenable case is when $X = \mathbb{R}$. Here, the 'spaces' X_t are just horizontal grid lines (each a copy of the real line). We illustrate this in Figure 3.2 with two real-valued fluents $x[t] = t$ and $y = x^2$. They have the same indexing interval I so they can be superimposed on the same diagram.

As t traverses I from the bottom to the top of the diagram, both x and y vary their values along each respective gridline $\mathbb{R}_t$. Fluent x increases at a uniform rate as t traverses I, but y initially decreases (moving leftwards in the diagram), before increasing (moving rightwards) from $t = 0$ onwards.

Both fluents in this example are continuous, but this isn't necessarily the case in general. Later on, we'll define fluent continuity in a way that mirrors the intuition that the corresponding plot is one that can be drawn without lifting the pen off the page.

Chapter 4

Properties of a Fluent

4.1 Change across an Interval

Because an indexing interval $[T_1, T_2] = I$ is finite for a fluent $x[t]$ defined over $t \in I$, it makes sense to talk of the change in x across I. We denote this as 'Δx across I', where $\Delta x = x[T_2] - x[T_1]$; or more compactly as $\Delta x|_I$. The latter notation applies naturally to $x|_J$ where x is restricted to a non-trivial closed sub-interval $J \subseteq I$.

In what follows, we will frequently wish to compare the change in fluent value across I with the length of the interval I. We therefore use the visually similar notation ΔI for this, where $\Delta I = T_2 - T_1$.

An indexing sub-interval $J = [t_1, t_2]$ will always have $t_1 < t_2$, so we can be sure that $\Delta J > 0$. This isn't the case for $\Delta x|_J$. For example, if the fluent $x[t] = t^2$ is defined for $t \in [-1, +1] = I$, then $\Delta x|_J$ could be positive, negative or zero, depending on the choice of sub-interval $J \subseteq I$.

Remark 4.1. Notice that for a fixed indexing interval I, Δx across I acts as a real-valued operator on fluent x. It's linear: ie. for any other fluent y defined on I and any $a \in \mathbb{R}$, $\Delta(x + y) = \Delta x + \Delta y$ across I and $\Delta(ax) = a\,\Delta x$. It also satisfies $\Delta(x + b) = \Delta x$ for any $b \in \mathbb{R}$, a form of translational invariance.

Both linearity and translational invariance are preserved throughout each of the constructions that follow in forming the fluxion and function derivative.

When we have $I = [T_1, T_2]$, then the same constructions carry forward their own versions of the following equivalent formulae for the change across I of the product of x and y:

$$\begin{aligned}\Delta(xy) &= y[T_2]\,\Delta x + x[T_1]\,\Delta y \\ &= y[T_1]\,\Delta x + x[T_2]\,\Delta y.\end{aligned}$$

4.2 Opposite of a Fluent

The opposite of a fluent can help in providing dualities when exploring properties of fluents or in constructing proofs (see Definition 3.3). We examine some key relations, which are preserved in a similar form in subsequent constructions.

For example, if w is the opposite of fluent x, with respective indexing intervals $H = [-T_2, -T_1]$ for w and $I = [T_1, T_2]$ for x, then $\Delta w|_H = -\Delta x|_I$, a duality in sign.

Note, however, that $\Delta H = \Delta I$ and in particular $\Delta H > 0$.

The following lemma summarises how constructing an opposite of a fluent interacts with more general fluent operations.

Lemma 4.1. *Suppose $a \in \mathbb{R}$ is a constant; and that x and y are real-valued fluents defined on a common indexing interval I. Then the following basic relations hold.*

1. $\operatorname{opp}(\operatorname{opp} x) = x$.
2. $\Delta(\operatorname{opp} I) = \Delta I > 0$
3. $\Delta\left(\operatorname{opp}\ x|_I\right) = -\Delta x|_I$
4. $\operatorname{opp}(x + a) = (\operatorname{opp} x) + a$.
5. $\operatorname{opp}(ax) = a \operatorname{opp} x$.
6. $\operatorname{opp}(x + y) = \operatorname{opp} x + \operatorname{opp} y$.
7. $\operatorname{opp}(xy) = (\operatorname{opp} x)(\operatorname{opp} y)$.

Proof. These follow directly from the relevant definitions. □

4.3 Introducing Flow Velocity

4.3.1 Case Study

We will often wish to find a formal definition of a conceptual property of a fluent. Typically, this conceptual property will have some kind of physical or intuitive meaning, which implies a testable condition on the fluent involved. Many of the definitions here and in the next chapter arise from the interplay between intuition and the appropriate testable conditions.

For example, we might equate a fluent $x[t]$ over an indexing interval I with the displacement of a particle at physical time $t \in I$ as we traverse I from start to finish. We then might ask whether it travels at a constant velocity c as t progresses.

A simple test might then be whether $\Delta x|_I = c\,\Delta I$ across I. If the test fails, then our constant velocity hypothesis will be falsified.

We can think of the condition $\Delta x|_I = c\,\Delta I$ as a property of the interval I. It's necessarily implied by x having constant velocity c on I with respect to t, but on its own it isn't sufficient to deduce that x's velocity really is constant over the whole of I. It just tells us that the *mean* velocity is c: it may be, for example, that x changes by the whole of $\Delta x|_I$ in the first half of the time interval, but is stationary (so doesn't change at all) in the second half. How might we find a sufficient condition?

A pattern emerges for this case and for other fluent properties we'll naturally encounter in this chapter. We note that if x has constant velocity c over interval I, then we expect the same property to be true of any indexing sub-interval $J \subseteq I$, so that $\Delta x|_J = c\,\Delta J$. We might regard J as *inheriting* this property from I. Applying this relationship to every such J provides a much stronger set of conditions, which prove to be sufficient to establish that x really does have constant velocity c.

Remark 4.2. We find that our definitions can be expressed in terms of "hereditary properties" within a given "interval class", conforming to the heuristic framework in Section 4.5. Rather than start with the framework straightaway, for now we'll simply remark when a definition uses one of its heuristics.

4.3.2 Uniform Flow Velocity

The previous discussion leads us to the following definition.

Definition 4.1. A fluent x has a *uniform flow velocity* c on an indexing interval I iff for each non-trivial closed sub-interval $J \subseteq I$, we have $\Delta x|_J = c\,\Delta J$.

Remark 4.3. If we set $w = \operatorname{opp} x$ from this definition, then we can quickly confirm that w will be a fluent with uniform flow velocity $-c$, in line with what we might intuitively expect. It has the same flow speed magnitude as x, but moves through the values of x in the opposite direction as we traverse each fluent's respective indexing interval.

Its own indexing interval is $\operatorname{opp} I$.

Remark 4.4. This definition conforms to Heuristic 4.1 below, with characteristic interval class $\mathcal{P}_\iota I$ and characteristic condition $\Delta x|_J = c\,\Delta J$.

The following result may be a more familiar characterisation of uniform velocity.

Theorem 4.2. *Suppose the fluent* $x[t]$ *is defined on an indexing interval* I *and passes through* $x_0 = x[t_0]$ *when* $t = t_0 \in I$. *Then* $x[t]$ *has uniform flow velocity* c *iff for each* $t \in I$, *it satisfies the relation* $x[t] - x_0 = c\,(t - t_0)$.

Proof. The relation holds trivially when $t = t_0$.

Given $x[t]$ with $x_0 = x[t_0]$, choose $t \in I$ with $t_0 < t$ (if any) and form the non-trivial sub-interval $J = [t_0, t]$. Then $\Delta x|_J = c\,\Delta J$ implies $x[t] - x_0 = c\,(t - t_0)$.

Similarly, $t < t_0$ implies $x_0 - x[t] = c\,(t_0 - t)$ and hence $x[t] - x_0 = c\,(t - t_0)$.

Hence the relationship holds for all $t \in I$. □

A simple consequence is that if a variable x takes values along a non-trivial closed interval K, then we can always express it as a fluent over a suitable indexing interval I so that it has a uniform non-zero flow velocity $c > 0$. (There are certain constraints, such as $c\,\Delta I \leq \Delta K$.) In particular, we can do this around any point $x_0 \in K^o$.

4.4 Continuity

4.4.1 Preamble

One approach to defining continuity is to contrast it with having a sudden jump at a point. Intuitively, a fluent $x[t]$ will be continuous at $x_0 = x[t_0]$ if there are no sudden jumps forwards or backwards in x as we pass through time t_0. We now try to capture this idea.

When we have a fluent $x[t]$ over an indexing interval $I = [t_1, t_2]$, we can think of the time-like parameter t as continuously advancing from the value t_1 to the larger value t_2. While that happens, it's possible that the value of the fluent x itself will not be changing continuously from one moment to another. For example, it may be that a $x[t]$ involves a sudden jump or goes back on itself or always stays at the same value (which has no change at all).

Example 4.1. As a first example, consider the stepped fluent $h_{\text{STEP}}[t]$, where:

$$h_{\text{STEP}}[t] = \begin{cases} 0 & \text{if } t < 0; \\ +1 & \text{if } t \geq 0. \end{cases}$$

There is a jump forwards from $h_{\text{STEP}}[t] = 0$ for $t < 0$ to $h_{\text{STEP}}[t] = +1$ for $t \geq 0$ when going through $t = 0$. The fluent's evaluation function is the Heaviside step function $\mathrm{H}(t)$[7], ie. $h_{\text{STEP}}[t] \equiv \mathrm{H}(t)$.

Example 4.2. Now consider $x_{\text{SPIKE}}[t]$ with:

$$x_{\text{SPIKE}}[t] = \begin{cases} 0 & \text{if } t < 0; \\ 1 & \text{if } t = 0; \\ 0 & \text{if } t > 0. \end{cases}$$

In this case, there is a clear jump forwards and then immediately backwards again when going through $t = 0$.

Example 4.3. For a more complicated case, consider:

$$x_{\mathrm{SNAKE}}[t] = \begin{cases} t \cos\frac{\pi}{t} & \text{if } t \neq 0; \\ 0 & \text{if } t = 0. \end{cases}$$

This is well-defined, has no obvious jumps (even at $t = 0$), but should we consider it continuous?

In our first two examples, the jump occurs at the single parameter value $t = 0$, with continuity elsewhere. In many applications, the second example of x_{SPIKE} would have no physical meaning (and we might be able to eliminate the discontinuity by setting $x_{\mathrm{SPIKE}}[0] = 0$).

It may not be clear whether continuity should apply along a whole interval (as with uniform flow velocity) or if we can define it at a single point. Because continuity is complementary to the existence of an instantaneous jump in value, we'll define it here for a given parametric point $t = t_0$ (which is consistent with the classical definition of continuity of a function).

One way to arrive at a definition of continuity of a fluent $x[t]$ is to think about how we might test for a jump from $x_0 = x[t_0]$ to at least some new value $b > x_0$ as we pass through t_0. In our first example it would be enough to show there's a jump in h_{STEP} from $h_{\mathrm{STEP}}[t] = 0$ at any $t < 0$ to a value greater than or equal to some $b > 0$, eg. $b = \frac{1}{2}$. We could have chosen any value of $b > 0$; and it's obvious practically that we'll require $b \leq 1$.

A naive test might be to choose a smallish interval of time $I = [T_1, T_2]$ straddling t_0, then look out for $x[t] \geq b$ as t goes from T_1 to T_2. It needs an interval of some form, because there's no such thing as a unique point just before or just after t_0 for testing. For example, we might choose $I = [t_0 - 10^{-6}, t_0 + 10^{-6}]$.

But, as for uniform velocity, no single interval is sufficient for a conclusive proof that x makes the jump as we pass through x_0, because the jump might be somewhere just before or just afterwards. Continuing with the last example, it may be that $x[t] = x_0$ when $t \in [t_0 - 10^{-9}, t_0 + 10^{-9}]$ and $x[t] = b$ elsewhere within I.

This suggests a sufficient condition for a jump to at least b to exist might be that for every J straddling t_0 there's at least one value $t \in J$ with $x[t] \geq b$. Continuity requires there to be no such jumps. In other words, we will be interested in the opposite condition that values of x will stay less than any given $b > x_0$. We formalise this in the next section.

4.4.2 Within-Bounds Continuity

Definition 4.2. The fluent x with indexing interval I is *strictly bounded above by* $b \in \mathbb{R}$ *through* $t_0 \in I^o$ iff for some closed sub-interval $J \subseteq I$ straddling t_0 we have $x|_J < b$.

I'll write this simply as '$x < b$ through t_0'.

(Recall that $x|_J < b$ means that for all $t \in J$ we have $x[t] < b$.)

Remark 4.5. This definition restricts the choice of t_0 to the interior of the indexing interval I to simplify the description of eligible intervals to those straddling t_0.

Fluent continuity can also make intuitive sense through either endpoint of I, eg. when there is no immediate jump leaving $t_0 = \min I$. There are other equivalent conditions for within-bounds continuity which can be extended more naturally to the whole of the interval I. (For example, saying the fluent x with indexing interval I is strictly bounded above by $b \in \mathbb{R}$ through $t_0 \in I^o$ iff for some closed sub-interval $H \subseteq \mathbb{R}$ straddling t_0 we have $x|_{H \cap I} < b$.)

A similar situation occurs later on in the analysis of non-uniform flow velocity.

Remark 4.6. Definition 4.2 conforms to Heuristic 4.2 (the dual form) below, where $\bar{P}(I)$ iff fluent $x[t]$ is strictly bounded above by b through $t_0 \in I^o$. We expect its negation to be hereditary, where $\neg\bar{P}(J)$ holds iff $x|_J$ jumps to b or above through t_0. The characteristic class is $\mathcal{P}_\iota I\big|_{t_0}^{\pm}$ of all sub-intervals $J \subseteq I$ straddling t_0, and the sufficing characteristic condition $\bar{Q}(J)$ holds iff $x|_J < b$. .

In this case, we might well expect $\bar{P}(J)$ to be a hereditary property as well, so we might consider applying Heuristic 4.1 (the base form). This strategy gets us no closer to a definition, because $\bar{Q}(I)$ logically implies $\bar{Q}(J)$ for all $J \subseteq I$ so the heuristic is neutralised: it can add no new content.

The fluents $-h_{\text{STEP}}$ and $\pm x_{\text{SPIKE}}$ from our examples have jumps backwards. This suggests the following obvious dual relation.

Definition 4.3. The fluent x with indexing interval I is *strictly bounded below by* $a \in \mathbb{R}$ *through* $t_0 \in I^o$ iff for some non-trivial closed sub-interval $J \subseteq I$ straddling t_0 we have $x|_J > a$ (ie. for all $t \in J$ we have $x[t] > a$).

I'll write this as '$a < x$ through t_0'.

This now sets us up to define continuity of $x[t]$ in terms of the lower and upper bounds on x.

Definition 4.4. If the fluent x has indexing interval I straddling t_0, then it is *within-bounds continuous through* t_0 iff x is strictly bounded below by a through t_0 for every $a < x_0$ and is strictly bounded above by b through t_0 for every $b > x_0$, where $x_0 = x[t_0]$.

We might write this condition as $\forall a < x_0\,(a < x$ through $t_0)$ and likewise $\forall b > x_0\,(x < b$ through $t_0)$.

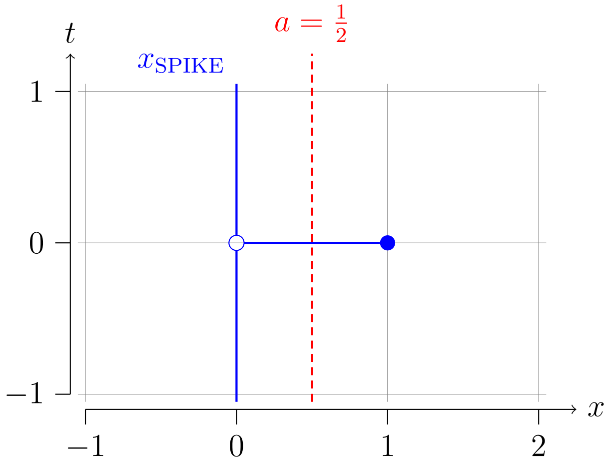

Figure 4.1: The fluent x_{SPIKE} with indexing interval $I \supseteq [-1, +1]$. The attempted lower bound $a = \frac{1}{2}$ satisfies $a < x_{\text{SPIKE}}[t]$ when $t = 0$, but fails for all $t \neq 0$, indicating a jump in value through $x_{\text{SPIKE}}[0]$.

Example 4.4. Return to the fluent x_{SPIKE} in Example 4.2 with $x_{\text{SPIKE}}[0] = 1$ and $x_{\text{SPIKE}}[t] = 0$ when $t \neq 0$. The condition fails for eg. $a = \frac{1}{2} < x_{\text{SPIKE}}[0]$ at both $x_{\text{SPIKE}}[t_1]$ and $x_{\text{SPIKE}}[t_2]$ for any non-trivial closed interval $[t_1, t_2]$ straddling $t_0 = 0$; so $x_{\text{SPIKE}}[t]$ in this example isn't continuous. (See Figure 4.1.)

Example 4.5. Is fluent $x_{\text{SNAKE}}[t]$ from Example 4.3 continuous through $t_0 = 0$?

Given $b > x_{\text{SNAKE}}[0] = 0$, set $J^\ddagger = [-\frac{1}{2}b, +\frac{1}{2}b]$, which straddles $t_0 = 0$. Then for all $t \in J^\ddagger$, $x_{\text{SNAKE}}[t] \leq \frac{1}{2}b < b$; hence $x_{\text{SNAKE}}|_{J^\ddagger} < b$ through $t_0 = 0$.

A dual argument with $J^\dagger = [-\frac{1}{2}a, +\frac{1}{2}a]$ shows $a < x_{\text{SNAKE}}|_{J^\dagger}$ through $t_0 = 0$ for every $a < x_{\text{SNAKE}}[0]$. This is true for all such a and b. Therefore $x_{\text{SNAKE}}[0]$ is within-bounds continuous through $t_0 = 0$.

Corollary 4.3. *If the fluent $x = (\mathbb{R}, U, I, \nu)$ has indexing interval I straddling t_0, then it's within-bounds continuous through t_0 iff for every finite non-trivial closed sub-interval $K \subseteq U$ straddling $x_0 = x[t_0]$, there exists a finite non-trivial closed sub-interval $H \subseteq I$ straddling t_0, where $x[t] \in K$ for every $t \in H$ (or equivalently $\nu(H) \subseteq K$).*

Proof. The proof proceeds by combining the definitions with the bounds $\min K$ and $\max K$.

Assume x is within-bounds continuous through t_0 and the finite non-trivial closed sub-interval $K \subseteq U$ straddles $x_0 = x[t_0]$, so $\min K < x_0 < \max K$. By the definition of within-bounds continuity, there will be some $J^\dagger$ straddling t_0 with $\min K < x[t]$ for all $t \in J^\dagger$; so $\min K < x|_{J^\dagger}$.

Similarly, we can find $J^\ddagger$ straddling t_0 with $x|_{J^\ddagger} < \max K$. Set $H = J^\dagger \cap J^\ddagger$; then this has the properties required by the theorem.

For the converse, suppose $a < x_0$ and $b > x_0$: we wish to find $J^\dagger$ and $J^\ddagger$ straddling t_0, where $a < x|_{J^\dagger}$ and $x|_{J^\ddagger} < b$. Set $K = [\frac{1}{2}(a + x_0), \frac{1}{2}(x_0 + b)]$, which straddles x_0; and $a < K < b$. By the converse hypothesis, we can find a finite closed sub-interval H straddling t_0 with $\nu(H) \subseteq K$. Setting $J^\dagger = J^\ddagger = H$ gives the desired result. ☐

Remark 4.7. The definition of within-bounds continuity can be couched in terms of spot values of $x[t]$ near $t = t_0$. It then looks very similar to the historic definition of ε-δ continuity by Karl Weierstrass (when applied in the natural way to a fluent).

Both approaches effectively bracket the value of $x[t]$ by taking a small enough interval around t_0. In the ε-δ approach, this is an open interval having the form $J^o =]t_0 - \delta, t_0 + \delta[$; which is symmetric about $t = t_0$, and with a half-length equal to $\delta > 0$. It's straightforward to prove that the two definitions are logically equivalent.

The interval-based condition in the corollary is very similar to the topological definition of continuity in terms of neighbourhoods.

4.5 Heuristic Framework

(This section isn't essential—it can be skipped if your overriding interest is the way fluxions can be used to define differential calculus.)

4.5.1 Interval Classes

This chapter motivates some unifying definitions, which largely have heuristic value. As such, they provide a general setting for defining fluent properties such as uniform flow velocity; they aren't, however, essential to the ways those properties are used. They're added here for interest only.

Definition 4.5. An *interval class* $\mathcal{C}$ is simply a set of real intervals.

More informally, we expect any interval $J \in \mathcal{C}$ to satisfy some condition expressible in the language of intervals and real numbers. For example, because we will usually be dealing with choices of indexing intervals for a fluent, the interval classes used here only include finite non-trivial closed intervals.

Definition 4.6. If I is a finite non-trivial closed interval straddling a point $t_0 \in I^o$, define the following three interval classes:

- $J \in \mathcal{P}_\iota I$ iff J is any non-trivial closed sub-interval of I;

- $J \in \mathcal{P}_\iota I\big|_{t_0}$ iff J is any non-trivial closed sub-interval of I and $t_0 \in J$;
- $J \in \mathcal{P}_\iota I\big|_{t_0}^{\pm}$ iff J is a closed sub-interval of I straddling t_0 (hence $t_0 \in J^o$ and J is non-trivial).

In this definition, the ι subscripts are a reminder that the classes involve indexing intervals, rather than more general subsets.

Corollary 4.4. $\mathcal{P}_\iota I\big|_{t_0}^{\pm} \subseteq \mathcal{P}_\iota I\big|_{t_0} \subseteq \mathcal{P}_\iota I$

Proof. The proof follows directly from the definitions. □

Example 4.6. When $I = [-2, 2]$ and $J = [0, 2]$:

- $I \in \mathcal{P}_\iota I$
- $J \in \mathcal{P}_\iota I\big|_0$
- $J \notin \mathcal{P}_\iota I\big|_0^{\pm}$
- $J \in \mathcal{P}_\iota I\big|_{1/2}^{\pm}$

4.5.2 Hereditary Properties

We refer interchangeably to a $\{\text{true}, \text{false}\}$ interval property P and the corresponding predicate function $P\colon \mathcal{C} \to \{\text{true}, \text{false}\}$ on some interval class $\mathcal{C}$. When J is a specific interval, then $P(J)$ is the proposition that J has property P.

Definition 4.7. A $\{\text{true}, \text{false}\}$ property (or a unary predicate) of intervals is *hereditary* within an interval class $\mathcal{C}$ iff whenever it holds for an interval J in $\mathcal{C}$ it also holds for every sub-interval $G \subseteq J$ which is in $\mathcal{C}$.

Remark 4.8. Only truth is inherited, not falsehood: a hereditary property may fail for J but hold for a sub-interval $G \subseteq J$ in $\mathcal{C}$ (and hence hold for each of G's sub-intervals).

Example 4.7. Put $I = [0, 1]$, $J = [0, \frac{3}{4}]$ and $G = [0, \frac{1}{4}]$. The property $P(H) \equiv \Delta H < \frac{1}{2}$ is hereditary for $H \in \mathcal{P}_\iota I$ (and satisfied by G). This example also illustrates how truth is inherited, but not falsehood: $\neg(\Delta H < \frac{1}{2})$ (equivalently $\Delta H \geq \frac{1}{2}$) is satisfied by I, but isn't inherited by G, even though it holds for J.

Example 4.8. Suppose fluent x has indexing interval $I = [0, 1]$. The property $\Delta x|_H < \frac{1}{4}$ may or may not be hereditary for $H \in \mathcal{P}_\iota I$. It's hereditary if $x[t] = t^2$; but not if $x[t] = (t - \frac{1}{2})^2$ (consider $H = [0, \frac{1}{2}] \subseteq I$).

$$\begin{tikzcd}
I \in \mathcal{C} \arrow[d, "\supseteq"'] & P(I) \arrow[r, Rightarrow] \arrow[d, Rightarrow] \arrow[dr, Rightarrow, "\forall J \in \mathcal{C} \cap \mathcal{P}\, I"] & Q(I) \\
J \in \mathcal{C} & P(J) \arrow[r, Rightarrow] & Q(J)
\end{tikzcd}$$

Figure 4.2: The dependencies in Corollary 4.5, with sub-intervals $J \subseteq I$ in the interval class $\mathcal{C}$. Inheritance means that $P(I)$ implies $P(J)$; and by hypothesis $P(I)$ implies $Q(I)$ and $P(J)$ implies $Q(J)$. Then for all $J \subseteq I$ in $\mathcal{C}$, we deduce that $Q(J)$ holds.

The following corollary motivates two useful heuristic principles in defining certain properties of fluents.

Corollary 4.5. *Suppose $\mathcal{C}$ is an interval class and that $P(J)$ represents the proposition 'interval J has property P' for $J \in \mathcal{C}$ (so P acts as a unary predicate).*

Suppose also:

- *property P is hereditary within the interval class $\mathcal{C}$; and*
- *for all J in $\mathcal{C}$, $P(J)$ implies $Q(J)$, where $Q(J)$ corresponds to an interval property Q on $J \in \mathcal{C}$.*

Then, for any interval $I \in \mathcal{C}$, $P(I)$ implies that condition $Q(J)$ holds for all sub-intervals $J \subseteq I$ with $J \in \mathcal{C}$.

Proof. By hypothesis, assume $P(I)$ holds and we have $I \in \mathcal{C}$.

Take a sub-interval $J \subseteq I$ with $J \in \mathcal{C}$. Property P is hereditary within $\mathcal{C}$, so $P(J)$ must hold.

But $P(J)$ implies $Q(J)$ for $J \in \mathcal{C}$. Hence $Q(J)$ holds, and this must be true for every such J. □

A hereditary property may be either formal or conceptual. By a conceptual property I mean one which is intuitive (a mental construct), or informal in some other way (perhaps founded empirically). When we have a conceptual property, we might intuitively expect it to be inherited by any suitable sub-interval. In this case, we might be categorising it as hereditary in line with our intuition, rather than formally deducing it must be. We show how we can then consider this insight to see if leads to an appropriate formalisation. Of course, once we fix on a formal definition for this property, we can check deductively that the definition really does imply it's hereditary in the way we'd expect.

Remark 4.9. In talking of intuition, I'm implicitly making a distinction between mathematical intuition and scientific induction, even when a mathematical concept is founded on empirical observation. The driving forces, constraints and verification practices of mathematics and science are somewhat different from each other.

This is particularly true when examining and explaining the nature of the continuum, which is an essential ingredient of differential calculus. Mathematicians will tend to regard the continuum as indefinitely divisible (and more besides); while physicists may be more circumspect, especially when they consider the extent to which matter is indefinitely divisible.

We now have enough machinery to identify two equivalent heuristic principles, which give rise to a number of different formal definitions in this chapter and the next.

Typically, we will have a fluent (or fluent continuation) x, which can be defined on any member $J \in \mathcal{C}$ of an interval class $\mathcal{C}$ of finite non-trivial closed intervals. Stretching our notation slightly, we can regard a $\{\text{true}, \text{false}\}$ fluent property $P(x|_J)$ for any $J \in \mathcal{C}$ as an interval property $P(J)$. And we can make a similar move if a relational predicate $P(x|_J, y|_J)$ compares two fluents x and y on the same indexing interval J.

In this way, what is ostensibly a fluent property is abstracted away to become a property of intervals within a specific interval class. This then opens the way to examining whether we expect the property to be hereditary or not. Although not strictly required, it may be helpful to think of the intervals involved in the following two heuristics as non-trivial, finite and closed (so they can legitimately index a fluent).

Heuristic Principle 4.1 (Base form)**.** Suppose $\mathcal{C}$ is an interval class. We have identified a conceptual property $P(J)$ that can be applied to any interval $J \in \mathcal{C}$.

Suppose also:

- property P is conceptually hereditary within $\mathcal{C}$; and
- for all J in $\mathcal{C}$, $P(J)$ informally implies $Q(J)$, where $Q(J)$ itself is a formal condition on J.

We require a formal definition for $P(J)$ in terms of Q such that $P(J) \Rightarrow Q(J)$; but we can't express $P(J)$ from $Q(J)$ alone. (Typically, $Q(J)$ may be necessary but not sufficient.)

If class $\mathcal{C}$ is sufficiently wide and property Q is suitably characteristic of P, then we may be able to define P as follows:

> "property $P(I)$ holds on $I \in \mathcal{C}$ iff condition $Q(J)$ holds for all subintervals $J \in \mathcal{C}$ with $J \subseteq I$."

If $\mathcal{C}$ and Q can be found to make the above strategy successful, then we call $\mathcal{C}$ a *characteristic interval class* for property P, and Q a *characteristic condition.*

Example 4.9. The definition of uniform flow velocity in Section 4.3.2 mirrors the constant velocity case study preceding it. The conceptual property $P(I)$ applied to fluent x is that of having uniform flow velocity (constant velocity) c over the indexing interval I; the characteristic interval class is $\mathcal{P}_\iota I$; and the characteristic condition $Q(J)$ is $\Delta x|_J = c\,\Delta J$ across J.

Of course, we can't prove this heuristic: we can only gather evidence of it working in practice as we proceed. We can also verify that $P(I)$ as defined in terms of Q can now be proved formally to be hereditary and does indeed imply $Q(I)$.

Corollary 4.6. *Suppose $\mathcal{C}$ is a fixed interval class of finite non-trivial closed intervals; and $Q(J)$ is a predicate defined for all $J \in \mathcal{C}$.*

Now assume, in line with Heuristic 4.1, we have a definition of the following form: the property $P(I)$ holds on $I \in \mathcal{C}$ iff $Q(J)$ holds for all sub-intervals $J \in \mathcal{C}$ with $J \subseteq I$.

Then:

- *P is hereditary within $\mathcal{C}$; and*
- *for all J in $\mathcal{C}$, $P(J)$ implies $Q(J)$.*

Proof. Assume $P(I)$ for $I \in \mathcal{C}$. Take $J \subseteq I$ with $J \in \mathcal{C}$. Then for each $G \subseteq J$ with $G \in \mathcal{C}$, we have $G \subseteq I$; hence $Q(G)$. But this is the condition for $P(J)$ to hold; hence P is hereditary in $\mathcal{C}$.

Now assume $P(J)$ holds for $J \in \mathcal{C}$. Then $Q(J)$ holds trivially, because $J \subseteq J$. ☐

Remark 4.10. For the heuristic to be effective, we need to be adding information about P in stating that we expect it to be hereditary: in other words, it loses its effectiveness if the quality of being hereditary is formally deducible from existing mathematical theory. (Arguably, this illustrates the importance of intuition or non-formal reasoning in expanding mathematical content.)

Similarly, the conclusion of Heuristic 4.1 must also be adding information. For example, the power of the heuristic would be neutralised if from $Q(I)$ we could already formally deduce that $Q(J)$ holds for all sub-intervals $J \subseteq I$ with $J \in \mathcal{C}$. We saw an example of this when we discussed within-bounds continuity in Section 4.4.

We can express the situation that a property $\bar{P}$ inherits falsehood (rather than truth) within an interval class $\mathcal{C}$ by stating that $\neg\bar{P}$ (denoting 'not $\bar{P}$') is hereditary in $\mathcal{C}$. In this case, if $I, J \in \mathcal{C}$ and J is a sub-interval of I (so $J \subseteq I$ in $\mathcal{C}$), then

$\neg\bar{P}(I) \Rightarrow \neg\bar{P}(J)$; or equivalently $\bar{P}(J) \Rightarrow \bar{P}(I)$. In other words, $\bar{P}(J)$ bubbles up to all supersets $I \supseteq J$ in $\mathcal{C}$.

This leads to the following heuristic, logically equivalent to the previous formulation.

Heuristic Principle 4.2 (Dual form)**.** Suppose $\mathcal{C}$ is an interval class. We have identified a conceptual property $\bar{P}(J)$ that can be applied to any interval $J \in \mathcal{C}$.

Suppose also:

- we expect $\bar{P}(J)$ to informally imply $\bar{P}(I)$ for all supersets $I \supseteq J$ in $\mathcal{C}$ (ie. property $\neg\bar{P}$ is conceptually hereditary within $\mathcal{C}$); and
- for all J in $\mathcal{C}$, $\bar{Q}(J)$ informally implies $\bar{P}(J)$, where $\bar{Q}(J)$ itself is a formal condition on J.

We require a formal definition for $\bar{P}(I)$ in terms of $\bar{Q}$ such that $\bar{Q}(I) \Rightarrow \bar{P}(I)$; but we can't necessarily deduce $\bar{Q}(I)$ from $\bar{P}(I)$ itself. (Typically, $\bar{Q}(I)$ is sufficient but not necessary.)

If class $\mathcal{C}$ is sufficiently wide and property $\bar{Q}$ is suitably characteristic of $\bar{P}$, then we may be able to define $\bar{P}$ as follows:

> "property $\bar{P}(I)$ holds on $I \in \mathcal{C}$ iff condition $\bar{Q}(J)$ holds for some subinterval $J \subseteq I$ with $J \in \mathcal{C}$."

Proof. We can prove that this heuristic follows from the previous Heuristic 4.1 by setting $P = \neg\bar{P}$ and $Q = \neg\bar{Q}$. These fulfil the previous assumptions.

We therefore stipulate that fluent x has property $\neg\bar{P}(I)$ on I iff condition $\neg\bar{Q}(J)$ holds for all sub-intervals $J \subseteq I$ with $J \in \mathcal{C}$.

Negating both sides of the equivalence gives us the required formal definition for $\bar{P}(I)$, namely that fluent x has property $\bar{P}(I)$ on I iff $\bar{Q}(J)$ for some sub-interval $J \in \mathcal{C}$ with $J \subseteq I$. $\square$

If $\mathcal{C}$ and $\bar{Q}$ can be found to make the above strategy successful, then we again call $\mathcal{C}$ a *characteristic interval class* for property $\bar{P}$. We call $\bar{Q}$ a *sufficing characteristic condition*; and $\neg\bar{Q}$ a *characteristic counter-condition* (compare it with Q in the premises of Heuristic 4.1).

Remark 4.11. This dual form of the heuristic lends itself naturally to fluent continuations defined on a domain D. We may be able to say that a property P holds on domain D iff it holds for some indexing interval $I \subseteq D$ within an appropriate interval class. If a property seems blindly applicable to fluents and fluent continuations alike, then it can be a hint that Heuristic 4.2 might apply.

Several basic fluent properties involve one or other of these patterns, such as uniform velocity over an interval, within-bounds continuity and flow velocity comparison (the latter is described in Chapter 5).

Chapter 5

Non-uniform Flow Velocities

5.1 Flow Velocity on an Interval

Although our Definition 4.1 of uniform flow velocity exhibits much of the machinery we'll require, the concept of uniform flow on its own doesn't get us very far. Fluents, like their physical counterparts, can have flow velocities that vary with respect to their time-like parameters. (Of course, because the time parameter t is notional, this velocity is notional too, so we're proceeding by analogy.) For example, if $y = x^2$ and fluent $x[t]$ has uniform flow, then we can expect $y[t]$'s flow to be non-uniform, increasing with increasing t when $t > 0$, because y will be growing more quickly per unit change in t as t gets larger.

Quantifying a varying flow velocity at any given instant t may seem overambitious at this stage. Indeed, it isn't at all obvious that there always will be a unique well-defined value for flow velocity in all cases. Instead, we ask a simpler but still fundamental question about a fluent x that will ultimately lead us to defining the fluxion: what does it mean for x to be flowing faster or slower than a given velocity u or than another fluent y?

An appropriate modification of the preamble of Chapter 4 provides one approach: instead of testing $\Delta x|_I = c\,\Delta I$ for equality, we test for the desired inequality. Hence, we can readily adapt Definition 4.1 to say what it means for a fluent x to be flowing faster along its indexing interval than a given test velocity $u \in \mathbb{R}$. While this won't be enough for what we need for our ultimate objective, it provides a useful rehearsal for the next section, which addresses flow velocity through a given time-like parameter instant.

Definition 5.1. A fluent x *flows faster than* $u \in \mathbb{R}$ *on* an indexing interval I iff for each non-trivial closed sub-interval $J \subseteq I$, we have $\Delta x|_J > u\,\Delta J$.

Dually, x *flows slower than* $v \in \mathbb{R}$ *on* I iff for each non-trivial closed sub-interval $J \subseteq I$, we have $\Delta x|_J < v\,\Delta J$.

Remark 5.1. As with Definition 4.1, this definition conforms to Heuristic 4.1 (the base form). The characteristic interval class is again $\mathcal{P}_\iota I$; and the characteristic 'faster than' condition is now $\Delta x|_J > u\,\Delta J$ (in contrast to $\Delta x|_J = u\,\Delta J$ for Definition 4.1).

Example 5.1. Suppose $x[t] = |t|$ on indexing interval $I = [-2, +2]$. Then for any non-trivial closed sub-interval $J = [t_1, t_2] \subseteq I$, we have $\Delta J = t_2 - t_1 > 0$ and $-\Delta J \leq \Delta x|_J = |t_2| - |t_1| \leq \Delta J$

Hence for any $u < -1$ and $v > +1$, we must have $u\,\Delta J < \Delta x|_J < v\,\Delta J$; so the fluent x flows faster than u on I and slower than v.

Example 5.2. Fix the non-negative integer $N \in \mathbb{N}$, and consider the fluent $y_N[t] = t^2$ on an interval $I = [N, N+1]$. For any non-trivial closed sub-interval $J = [t_1, t_2] \subseteq I$, we can deduce that:

$$\begin{aligned}
\Delta y_N|_J &= t_2^2 - t_1^2 \\
&= (t_2 + t_1)(t_2 - t_1) \\
&= (t_2 + t_1)\,\Delta J \\
&> 2N\,\Delta J.
\end{aligned}$$

Hence y_N flows faster than $2N$ on I, with this lower bound on flow velocity increasing as N increases.

The next definition compares the flow velocity of two fluents x and y on an indexing interval I. It's a simple generalisation of Definition 5.1.

Definition 5.2. Suppose that fluents x and y share the same indexing interval I. Then *x flows faster than y on I* iff for each non-trivial closed sub-interval $J \subseteq I$, we have $\Delta x|_J > \Delta y|_J$, or equivalently $\Delta y|_J < \Delta x|_J$. We denote this as '$x \gtrdot y$ on I'.

Dually, *x flows slower than y on I* iff for each non-trivial closed sub-interval $J \subseteq I$, we have $\Delta x|_J < \Delta y|_J$. We then write '$x \lessdot y$ on I'.

Remark 5.2. This definition again conforms to Heuristic 4.1 (the base form) with characteristic interval class $\mathcal{P}_\iota I$. The characteristic 'faster than' condition is now $\Delta x|_J > \Delta y|_J$.

We can reconcile this definition with Definition 5.1 straightforwardly by introducing a fluent ψ_1 on I with unit flow velocity (hence $\Delta\psi_1|_J = \Delta J > 0$). This enables us to say that x flows faster than velocity u on I iff x flows faster than fluent $u\,\psi_1$ on I, written more compactly as $x \gtrdot u\psi_1$.

More specifically, let $\psi_1[t] = t$ be the fluent with unit uniform flow velocity, travelling through zero at zero time. (More strictly, we make ψ_1 to be a fluent continuation, which is defined along the whole of $\mathbb{R}$ and can be restricted to any required indexing interval.) Then we can define y to be a fluent with uniform flow velocity u by setting $y[t] = y_0 + u\,\psi_1[t - t_0]$ for some choice of y_0 and t_0; and therefore $\Delta y|_J = u\,\Delta J$.

Hence x flows faster than velocity u on I iff $x \gtrdot y$ there. In fact, we can simplify by noting that $\Delta y|_J = \Delta(u\psi_1|_J) = u\,\Delta\psi_1|_J$; hence x flows faster than velocity u on I iff $x \gtrdot u\psi_1$ on I. We use this as a convenient notation, as well as its counterparts $u\psi_1 \lessdot x$ and $x \lessdot v\psi_1$, etc.

Example 5.3. Suppose fluent x has uniform flow velocity c on indexing interval I. Then for any $v > c$, we have $x \lessdot v\psi_1$ on I.

To prove this, note that $\Delta x|_J = c\,\Delta J < v\,\Delta J = \Delta(v\psi_1|_J)$.

One could take this example further with the following lemma.

Lemma 5.1. *Fluent x has uniform flow velocity c on indexing interval I iff for all $u, v \in \mathbb{R}$ with $u < c < v$, we have $u\psi_1 \lessdot x \lessdot v\psi_1$ on I.*

Proof. This isn't a key result, so we just sketch a proof. The previous example indicates a proof starting from uniform flow.

For the converse, choose any indexing sub-interval $J \subseteq I$. Then for all $v > c$, we have $x \lessdot v\psi_1$ and hence $\Delta x|_J < v\,\Delta J$. Because $\Delta J > 0$, this then implies that $\Delta x|_J \leq c\,\Delta J$.

We can show similarly that $\Delta x|_J \geq c\,\Delta J$; and hence $\Delta x|_J = c\,\Delta J$. Because this is true for all such J, we conclude that x has uniform flow velocity c on I. □

Definition 5.3. The fluent x with indexing interval I has *positive flow velocity on I* iff $x \gtrdot 0$ on I. (In this context, '0' is the zero fluent, which has the value $0 \in \mathbb{R}$ for all $t \in I$.)

The next theorem helps to characterise the relation $x \lessdot y$ on an indexing interval I; and also helps to justify our choice of definitions. Any two of the three fluents x, y, z involved may be chosen freely, which then determines the third such that $y = x + z$. For example, if x and y are chosen, then the fluent $z = y - x$.

Theorem 5.2. *Suppose x, y and z are three fluents sharing an indexing interval I; and $y = x + z$ (so $z = y - x$). Then the following three statements are equivalent.*

1. *$x \lessdot y$ on I.*

2. *$z \gtrdot 0$ on I, ie. z has positive flow velocity on I.*

3. *The function $t \mapsto z[t]$ is strictly increasing on I.*

Proof. We prove each pair of implications involving Condition 2.

- $1 \implies 2$: $x \lessdot y$ on I implies that for all non-trivial closed $J \subseteq I$ we have:

  $$\Delta x < \Delta y \text{ on } J$$

  and hence:

  $$0 < \Delta y - \Delta x = \Delta z \text{ on } J$$

  Therefore $0 \lessdot z$ on I.

- $2 \implies 1$: The converse relation follows similarly.

- $2 \implies 3$: Choose $t_1, t_2 \in I$ with $t_1 < t_2$. Then $J = [t_1, t_2]$ is a non-trivial closed sub-interval of I.

  Now $0 \lessdot z$ on I implies that $0 < \Delta z|_J = z[t_2] - z[t_1]$; ie. $z[t_1] < z[t_2]$, which is the condition for the function $t \mapsto z[t]$ to be strictly increasing.

- $3 \implies 2$: Again, the converse relation follows fairly straightforwardly.

☐

Informally, this says that a fluent has positive flow velocity on its indexing interval when it is strictly increasing; and one fluent flows faster than another on their shared indexing interval when the gap between them is strictly increasing.

When a fluent x has uniform flow velocity c on an interval I, then we might reasonably expect that there is a meaningful instantaneous flow velocity at any time-like instant $t_0 \in I$, which will again be equal to c. Similarly, if fluent y flows faster than u on I, then for any non-trivial closed sub-interval J containing t_0, we know that $u\,\Delta J < \Delta y|_J$. This observation suggests we can meaningfully describe y as flowing faster than u through t_0 for any time-like instant $t_0 \in I$ (in fact, this will prove over-ambitious: we'll generally require $t_0 \in I^o$). It also provides us with a sanity check when we explore flow velocity through a given instant in the next section.

Remark 5.3. We might be tempted to turn this into a definition by stating that a fluent x defined on I flows faster than u through $t_0 \in I$ iff there is an indexing sub-interval $I^\dagger \subseteq I$ with $t_0 \in I^\dagger$ and x flows faster than u on $I^\dagger$. It produces a theory of fluxions and function derivatives which are necessarily continuous, while the definition in the next section accommodates more general discontinuous results.

The ingredient that makes this approach less general than how we'll be proceeding shortly is that the test condition for flow velocity through $t_0 \in I$ would include sub-intervals $J \subseteq I^\dagger$ that didn't contain t_0, putting constraints on the flow through time-like intervals separated from t_0 by a non-zero gap. The definition in the next section requires $t_0 \in J$ for its test intervals.

(In the language of interval classes, the next section works with characteristic interval class $\mathcal{P}_\iota I^\dagger\big|_{t_0}$, rather than the broader $\mathcal{P}_\iota I^\dagger$ advocated here.)

5.2 Flow Velocity through a time-like Instant

In the above discussion of flow velocity on an interval, we considered the possibility of comparing flow velocities through a time-like instant t_0. We now wish to define this formally.

Suppose fluent x has indexing interval I, which straddles a fixed time-like instant t_0. A sufficient condition for x to flow faster than velocity $u \in \mathbb{R}$ through t_0 seems to be that for any non-trivial closed sub-interval $J \subseteq I$ containing t_0, we have $\Delta x|_J > u\,\Delta J$; ie. for every such J, x travels further through $x[t_0]$ from its start to its end value than it would if it were travelling at constant velocity u. In that case, it can be convenient (if inelegant) to say 'x flows faster than u on I through t_0'.

Although this may be a sufficient condition for what we need, it isn't a necessary one: for example, the condition may fail for I itself, but not for a tighter interval around t_0. The next example shows how this might happen.

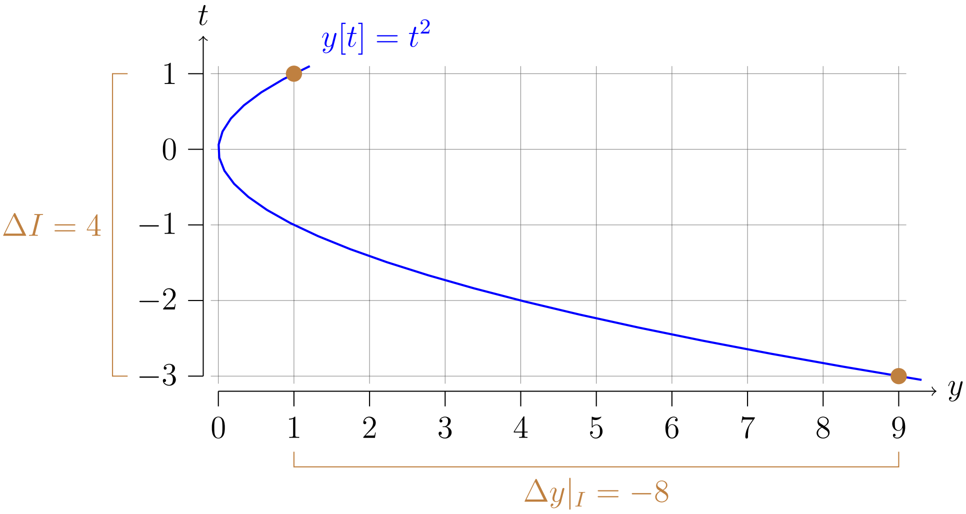

Figure 5.1: The fluent $y[t] = t^2$ in Example 5.4. As t traverses the interval $I = [-3, +1]$, the change in y is $\Delta y|_I = -8$. It is negative because the y-value undergoes a net movement from right to left in the diagram.

Example 5.4. Consider the fluent continuation $y[t] = t^2$ for $t \in \mathbb{R}$. We've seen previously how to deduce that $\Delta y|_J = (t_2 + t_1)\,\Delta J$ for any $J = [t_1, t_2]$.

Now fix an indexing interval $I = [-3, +1]$, which straddles $t_0 = 0$. The resulting fluent is $y|_I$, which can be restricted to the narrower sub-interval $H = [-1, +1] \subseteq I$. We show that y flows faster than -2 on H through $t_0 = 0$, but not on I itself.

For any $J = [t_1, t_2] \subseteq H$ we must have $t_2 > t_1 \geq -1$. Hence $-2\psi_1 \lessdot y$ on H; so (with obvious notation) we'd expect $-2\psi_1 \lessdot y$ through $t_0 = 0$.

We can easily show that this inequality fails for the indexing interval I itself, ie. that the condition $u\,\Delta I < \Delta y|_I$ fails for $u = -2$. Substituting $I = [-3, +1]$ to evaluate ΔI and $\Delta y|_I$, leads us to the falsehood $-8 < -8$ ※.

These considerations motivate the following definitions, the first supporting the second.

Definition 5.4. Suppose the fluent x has indexing interval I and $H \subseteq I$ is a non-trivial closed sub-interval straddling the time-like instant $t_0 \in I^o$ (hence, if we wish, we can form the restriction $x|_H$ of x to H).

Then x *flows faster than* u *on* H *through* t_0 iff for every non-trivial closed sub-interval $J \subseteq H$ containing t_0, we have $\Delta x|_J > u\,\Delta J$, or equivalently $u\,\Delta J < \Delta x|_J$.

Dually, x *flows slower than* v *on* H *through* t_0 iff for every non-trivial closed sub-interval $J \subseteq H$ containing t_0, we have $\Delta x|_J < v\,\Delta J$.

Definition 5.5. The fluent x on indexing interval I *flows faster than* velocity $u \in \mathbb{R}$ *through* the time-like instant $t_0 \in I^o$ iff it can be restricted to an indexing interval $I^\dagger \subseteq I$ straddling t_0 such that x flows faster than u on $I^\dagger$ through t_0, ie. for every non-trivial closed sub-interval $J \subseteq I^\dagger$ containing t_0, we have $\Delta x|_J > u\,\Delta J$.

We can then simply say 'x flows faster than u through t_0'.

Dually, it *flows slower than* v *through* t_0 iff for every such J within a restriction interval $I^\ddagger$ straddling t_0 we have $\Delta x|_J < v\,\Delta J$.

We can think of $I^\dagger$ and $I^\ddagger$ here as two (suitably chosen) restriction intervals. Generally, they needn't be equal. Nevertheless, we can assume equality without loss of generality by using the restriction interval $I^\dagger \cap I^\ddagger$

Remark 5.4. This is a two-tier definition, appealing to one of our heuristics at each tier.

The main definition conforms to Heuristic 4.2 (the dual form), with characteristic interval class $\mathcal{P}_\iota I\big|_{t_0}^{\pm}$. The sufficing characteristic condition $\bar{Q}(I^\dagger)$ for the 'flows faster' case is that x flows faster than u on $I^\dagger$ through t_0, ie. $\forall J \in \mathcal{P}_\iota I^\dagger\big|_{t_0} (\Delta x|_J > u\,\Delta J)$.

The definition of 'x flows faster than u on $I^\dagger$ through t_0' conforms to Heuristic 4.1 (the base form), with characteristic interval class $\mathcal{P}_\iota I^\dagger\big|_{t_0}$; and characteristic condition $\Delta x|_J > u\,\Delta J$.

The next definition compares the flow velocity of two fluents x and y through t_0, which is the natural generalisation of Definition 5.5. (We take the corresponding generalisation of Definition 5.4 as implied.)

Definition 5.6. Suppose that fluents x and y share the same indexing interval I. The fluent x *flows faster than* fluent y *through* the time-like instant $t_0 \in I^o$ iff it can be restricted to an indexing interval $I^\dagger$ straddling t_0 such that x flows faster than y on $I^\dagger$ through t_0.

In other words, for every non-trivial closed sub-interval $J \subseteq I^\dagger$ containing t_0, we then have $\Delta x|_J > \Delta y|_J$.

We denote this as '$x \gtrdot y$ through t_0'.

Dually, we can also say that y *flows slower than* x *through* t_0 and write '$y \lessdot x$ through t_0'.

Remark 5.5. This is again a two-tier definition, appealing to Heuristic 4.2 (the dual form) in the primary tier and then Heuristic 4.1 (the base form) in the secondary one.

You might wonder if x 'flowing faster through t_0' is truly independent of the indexing interval I (especially given Remark 5.7 below); and hence to what extent it's a property of the fluent about t_0 alone. The next theorem may give some reassurance.

Theorem 5.3. *Suppose $x|_I$ and $x|_K$ are restrictions of the same fluent (or fluent continuation) x to two indexing intervals I and K, each straddling t_0; and y is another fluent (or continuation) permitting the same indexing intervals as x.*

Then $x|_I$ flows faster than $y|_I$ through t_0 iff $x|_K$ flows faster than $y|_K$ through t_0.

Proof. Firstly, suppose $x|_I$ flows faster than $y|_I$ through t_0. We show the same is true of $x|_K$ and $y|_K$ as follows.

By hypothesis, we can find a non-trivial closed interval $I^\dagger \subseteq I$ straddling t_0 such that x flows faster than y on $I^\dagger$ through t_0.

Put $K^\dagger = I^\dagger \cap K$. Then $K^\dagger$ is closed and $t_0 \in K^\dagger$. In fact, $K^\dagger$ straddles t_0, since both $I^\dagger$ and K do. Viz. $K^\dagger$ is a non-trivial closed sub-interval of K straddling t_0.

Also, for any $J \in K^\dagger \subseteq I^\dagger$ with $t_0 \in J$, we must have $\Delta x|_J > \Delta y|_J$. And this just says that $x|_K$ flows faster than $y|_K$ on $K^\dagger$ through t_0.

Hence $x|_K$ flows faster than $y|_K$ through t_0, as required.

We've shown that if $x|_I$ flows faster than $y|_I$ through t_0, then $x|_K$ flows faster than $y|_K$ through t_0. The converse holds by symmetry. $\square$

Remark 5.6. Another way of thinking about this result is that if $P(J)$ is the property '$x|_J \lessdot y|_J$ through t_0', then both P and $\neg P$ are hereditary.

In other words, if $H \subseteq I \subseteq K$ are three indexing intervals straddling t_0, on which x and y are defined, and $x|_I \lessdot y|_I$ through t_0; then we can be assured that both $x|_H \lessdot y|_H$ through t_0, and also $x|_K \lessdot y|_K$ through t_0.

Echoing the previous section, we can reconcile this last Definition 5.6 with Definition 5.5 straightforwardly by setting $y = u\psi_1$. Here's the basic consistency check.

Lemma 5.4. *Suppose $u, v \in \mathbb{R}$ are velocity-like constants and I is an indexing interval straddling t_0. Then $u\psi_1 \lessdot v\psi_1$ through t_0 iff $u < v$.*

Proof. We first prove that $u < v$ implies $u\psi_1 \lessdot v\psi_1$ through t_0; and then prove its converse.

1. *Assume $u < v$*: Then, for any non-trivial closed sub-interval $J \subseteq I$ containing $t_0 \in I^o$, we must have $u\,\Delta J < v\,\Delta J$.

   But $\Delta J = \Delta\psi_1|_J$; and hence $u\psi_1 \lessdot v\psi_1$ on I through t_0; which implies $u\psi_1 \lessdot v\psi_1$ through t_0, as required.

2. *Assume $u\psi_1 \lessdot v\psi_1$ through t_0*: Then $u\,\Delta\psi_1|_J < v\,\Delta\psi_1|_J$ for any suitable non-trivial closed sub-interval J containing t_0.

   There's at least one such J, where necessarily $\Delta\psi_1|_J = \Delta J > 0$.

   We can therefore cancel $\Delta\psi_1|_J$ to get $u < v$.

This completes the proof. □

We can combine these results with the previous definitions to express some examples in terms of ψ_1.

Example 5.5. Suppose we have the fluent continuation $x[t] = |t|$. Then for any $u < -1$ and $v > +1$, $u\psi_1 \lessdot x \lessdot v\psi_1$ through any $t_0 \in \mathbb{R}$.

Example 5.6. For the stepped fluent h_{STEP} in Example 4.1, we always have $\Delta h_{\text{STEP}}|_J \geq 0$ for any non-trivial closed sub-interval $J \subseteq \mathbb{R}$. Hence $u\psi_1 \lessdot h_{\text{STEP}}$ through any $t_0 \in \mathbb{R}$ for any $u < 0$. In contrast, there is no $v \in \mathbb{R}$ with $h_{\text{STEP}} \lessdot v\psi_1$ through $t_0 = 0$.

5.3 Positive Flow Velocity

The possibility of comparing two flow velocities through a time-like instant t_0 provides an alternative definition of positive flow velocity. Conversely, we could have first defined the concept of positive flow velocity through a time-like instant t_0, then used it to define what we meant by the flow velocity comparison through t_0 between any two fluents.

Definition 5.7. The fluent x with indexing interval I has *positive flow velocity through* t_0 iff $x \gtrdot 0$ through t_0.

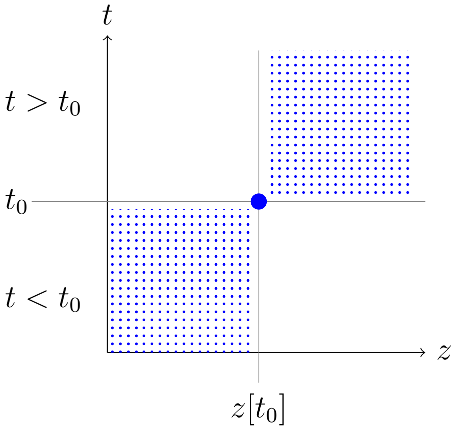

Figure 5.2: When $z \gtrdot 0$ through t_0, then $\nu_z(t_1) < \nu_z(t_0) < \nu_z(t_2)$ whenever $t_1 < t_0 < t_2$. The range of possibilities is represented by the two dotted regions.

As you might be anticipating from the previous section, the next theorem helps to characterise the relation '$x \lessdot y$ through t_0'; and throws more light on our choice of definitions. Any two of the three fluents x, y, z involved may be chosen freely, which then determines the third such that $y = x + z$.

Theorem 5.5. *Suppose x, y and z are three fluents sharing an indexing interval I straddling instant t_0; and $y = x + z$ (so $z = y - x$). Then the following three statements are equivalent.*

1. *$x \lessdot y$ through t_0.*

2. *$z \gtrdot 0$ through t_0, ie. z has positive flow velocity through t_0.*

3. *Define the evaluation function $\nu_z \colon I \to \mathbb{R}$ as the map $t \mapsto z[t]$. Then there's a non-trivial closed interval H straddling t_0, where for each pair of instants $t_1, t_2 \in H$ with $t_1 < t_0 < t_2$ we have $\nu_z(t_1) < \nu_z(t_0) < \nu_z(t_2)$.*

   In other words, the map $t \mapsto z[t]$ is strictly less than $z[t_0]$ for $t < t_0$ in H and strictly greater than $z[t_0]$ for $t > t_0$.

Proof. We again prove each pair of implications involving Condition 2.

- $1 \implies 2$: $x \lessdot y$ through t_0 implies that we can find a non-trivial closed sub-interval $H \subseteq I$ straddling t_0 such that for all non-trivial closed $J \subseteq H$ containing t_0 we have:

  $$\Delta x < \Delta y \text{ on } J$$

  and hence:

  $$0 < \Delta y - \Delta x = \Delta z \text{ on } J$$

Therefore $0 \lessdot z$ through t_0.

- $2 \implies 1$: The converse relation follows similarly.
- $2 \implies 3$: If z has positive flow velocity through t_0, there is a non-trivial closed sub-interval $H \subseteq I$ straddling t_0 where $z \gtrdot 0$ on H through t_0.

  Choose $t_1, t_2 \in H$ with $t_1 < t_0 < t_2$. Then $J = [t_1, t_0]$ is a non-trivial closed sub-interval of H for which $0 < \Delta z|_J = z[t_0] - z[t_1]$, ie. $\nu_z(t_1) < \nu_z(t_0)$.

  We show $\nu_z(t_0) < \nu_z(t_2)$ similarly, using the interval $J = [t_0, t_1]$.
- $3 \implies 2$: Again, the converse relation follows fairly straightforwardly.

□

Informally, this theorem says that a fluent z has positive flow velocity through t_0 iff for a suitably chosen indexing interval straddling t_0, the map $t \mapsto z[t]$ is strictly less than $z[t_0]$ for $t < t_0$ and strictly greater than $z[t_0]$ for $t > t_0$.

Similarly, a fluent y flows faster than another x through t_0 iff there's a suitable shared indexing interval H straddling t_0 on which the gap $z = y - x$ between them is initially strictly less than the gap $z[t_0] = y[t_0] - x[t_0]$ at t_0, then after t_0 strictly exceeds it. When $y[t_0] = x[t_0]$, so the two fluents coincide at t_0, then we can interpret this as saying that y overtakes x through $t = t_0$ as the time-like parameter t traverses indexing interval H.

We can use this theorem to show the extent to which $x \lessdot y$ on I corresponds to $x \lessdot y$ through each $t_0 \in I$. The correspondence is complicated by the status of the endpoints of I. We take the easiest direction first.

Theorem 5.6. *If x and y are two fluents with indexing interval I, and $x \lessdot y$ on I; then $x \lessdot y$ through each $t_0 \in I^o$.*

Proof. Assume $x \lessdot y$ on I, and choose any $t_0 \in I^o$.

We can simply take $I^\dagger = I$ in Definition 5.6, which qualifies as a non-trivial finite closed sub-interval of I straddling t_0.

Now choose any non-trivial finite closed sub-interval $J \subseteq I^\dagger$ containing t_0. We want to show that $\Delta x|_J < \Delta y|_J$.

But this is implied by our assumption that $x \lessdot y$ on I. Hence $x \lessdot y$ through t_0; and this is true of each such t_0. □

Theorem 5.5 helps with the converse via the following lemma.

Lemma 5.7. *Suppose that fluent z is defined on an indexing interval K and we restrict z to a non-trivial closed sub-interval $I \subseteq K^o$. Then $z \gtrdot 0$ through each $t_0 \in I$ implies that $z \gtrdot 0$ on I.*

Proof. Before proceeding, note that $z \gtrdot 0$ through t_0 makes sense when $t_0 \in I$ (including the end-points of I), because then $t_0 \in K^o$.

Now assume that $z \gtrdot 0$ through each $t_0 \in I$, and choose any non-trivial finite closed sub-interval $J \equiv [t_1, t_2] \subseteq I$, where $t_1 < t_2$. We want to show that $\Delta z|_J > 0$, ie. that $z[t_2] > z[t_1]$.

1. We start by defining a predicate $P(t)$ on J, where $P(t)$ is true iff $z[t_1] \leq z[t']$ for each $t' \in [t_1, t]$, ie. for each t' with $t_1 \leq t' \leq t$. Note that $P(t_1)$ holds.

2. P is a lower property on J, so Corollary 2.10 applies and P has a supremum $c = \sup P \in J$.

3. Now, $t' \in [t_1, c[\subseteq J$ implies $P(t')$ holds, by definition of the supremum; and hence $z[t_1] \leq z[t']$ for each $t' \in [t_1, c[$.

4. By hypothesis, $z \gtrdot 0$ through t_1, so by Theorem 5.5 there's a point $t_1^+ > t_1$ where $z[t'] > z[t_1]$ for each $t' \in]t_1, t_1^+]$ (ie. with $t_1 < t' \leq t_1^+$); hence $P(t_1^+)$ holds, and we must therefore have $t_1 < t_1^+ \leq c$.

5. We now show that $c = t_2$; since otherwise we can derive a contradiction as follows.

   Assume $c \neq t_2$, so we now have $t_1 < c < t_2$, ie. $c \in J^o$ and $z|_J \gtrdot 0$ through c makes sense and by hypothesis is true. Hence, we can find a closed sub-interval $G = [s_1, s_2] \subseteq J$ straddling c, where for each $s_1', s_2' \in G$ with $s_1' < c < s_2'$ we have $z[s_1'] < z[c] < z[s_2']$.

   In particular, taking $s_1 \equiv \min G \in [t_1, c[$, shows that $z[t_1] \leq z[s_1] < z[c]$; and therefore $P(c)$ holds.

   This in turn implies that $P(s_2)$ is true, where $s_2 = \max G$, because $z[t_1] \leq z[t']$ for each $t' \in [t_1, s_2]$. But $c < s_2$ implies that $P(s_2)$ must be false, and we have a contradiction ※. We must therefore have $t_2 = c = \sup P$.

6. We next apply Theorem 5.5 to the relation $z|_K \gtrdot 0$ through $t_2 = c \in I$ to find a corresponding interval $H \subseteq K$ with $\min H < t_2$ and, for all $t' \in H$ with $t' < t_2$, $z[t'] < z[t_2]$.

   Now consider $H^* = H \cap J$, for which $t_1 \leq \min H^* < t_2 = c$. Considering these inequalities within J shows that $z[t_1] \leq z[\min H^*] < z[t_2]$; and hence $\Delta z|_J > 0$.

This is true of any non-trivial finite closed sub-interval $J \subseteq I$. Therefore $z \gtrdot 0$ on I, as required. □

Theorem 5.8. *Suppose that x and y are two fluents sharing an indexing interval K and we restrict them to a non-trivial closed sub-interval $I \subseteq K^o$. Then $x \lessdot y$ through each $t_0 \in I$ implies that $x \lessdot y$ on I.*

Proof. This follows directly from the previous lemma by setting $y = x + z$. ☐

5.4 Basic Properties

A number of simple results follow from the previous definitions.

Lemma 5.9. *Suppose $a, b \in \mathbb{R}$; and that w, x, y, z are any four real-valued fluents defined on a common indexing interval I, which straddles $t_0 \in I^o$. Then the following basic relations hold for flow velocity through t_0.*

1. *(Duality 1) In general, $x \lessdot y$ iff $y \gtrdot x$.*
2. *(Duality 2) In general, $\operatorname{opp} x \lessdot \operatorname{opp} y$ iff $x \gtrdot y$.*
3. *(Transitivity) If $x \lessdot y$ and $y \lessdot z$, then $x \lessdot z$.*
4. *(Asymmetry) If $x \lessdot y$, then $\neg(y \lessdot x)$.*
5. *(Not reflexive) The above implies $\neg(x \lessdot x)$.*
6. *(Translational invariance 1) If $x \lessdot y$, then $x + a \lessdot y + a$.*
7. *(Scale invariance) If $0 < a$ and $x \lessdot y$, then $ax \lessdot ay$.*
8. *(Scale duality) If $a < 0$ and $x \lessdot y$, then $ay \lessdot ax$.*
9. *(Translational invariance 2) If $x \lessdot y$, then $x + w \lessdot y + w$.*
10. *Hence $x \lessdot y$ iff $0 \lessdot y - x$.*
11. *Hence if $0 < a < b$ and $0 \lessdot x$, then $ax \lessdot bx$.*
12. *If $x \lessdot y$ and $w \lessdot z$, then $x + w \lessdot y + z$.*

The obvious dual relations also apply.

Proof. The proof of each statement is straightforward, by expanding it according to the definitions. Note that if the pre-condition involves two comparisons of flow velocity, there will be two potentially different restriction intervals $I^\dagger$ and $I^\ddagger$ involved. The conclusion then holds for the restriction interval $I^* = I^\dagger \cap I^\ddagger$. Care is also needed in the way that an inequality will be reversed when multiplied by a negative number. ☐

Transitivity, asymmetry and non-reflexivity determine that $x \lessdot y$ is a strict partial order on x and y.

Note that if $x \gtrdot u\psi_1$ through t_0 and $u > w$, then Lemma 5.4 combined with transitivity implies that $x \gtrdot w\psi_1$. This says, as we'd intuitively expect, that if x flows faster than velocity u though t_0 and u exceeds w, then $x[t]$ flows faster than velocity w.

A dual relationship holds if x flows slower than u and u is less than w.

5.5 Commentary

(This section isn't essential—it can be skipped if your overriding interest is the way fluxions can be used to define differential calculus.)

5.5.1 Choices in these Definitions

Remark 5.7. It may not have been immediately apparent why we required the condition that $I^\dagger$ straddles t_0 in Definitions 5.5 and 5.6 (and similarly for $I^\ddagger$). Since $I^\dagger$ is a sub-interval of I, it is then necessarily the case that I straddles t_0 as well, ie. $t_0 \in I^o$.

One issue is that if we relax the condition, for example so that $t_0 \in I$ (rather than the stricter $t_0 \in I^o$), then inconsistencies can arise if I is a sub-interval of a larger indexing interval K, because there isn't a unique way to extend the fluent x from I to K. Notably, if t_0 is one of the end-points of I, we might then find that x flows faster than u through t_0 when restricted to I, but not when it's indexed by K.

Another issue occurs if we allow t_0 to be one of the endpoints of $I^\dagger$ in our definitions: transitivity breaks down, leading to unexpected results.

We find, for example, that if $x[t] = |t|$ on the interval $I = [-1, +1]$ which straddles $t_0 = 0$, and we're allowed to choose $I^\dagger = [-1, 0]$, then x flows slower than $-\frac{1}{2}$ through $t_0 = 0$, because for each $J \subseteq I^\dagger$ containing t_0 we have $\Delta x|_J = -\Delta J < -\frac{1}{2}\Delta J$. But then if we can also choose the restriction interval $I^\ddagger = [0, +1]$, we have the intuitively troubling result that x flows simultaneously faster than $+\frac{1}{2}$ through t_0.

The resulting chain of velocity relations through t_0 would be:

$$x \lessdot -\tfrac{1}{2}\psi_1 \lessdot +\tfrac{1}{2}\psi_1 \lessdot x$$

suggesting that $x \lessdot x$ ※.

Remark 5.8. There is a dual question in Definitions 5.5 and 5.6 on the need for each $J \subseteq I^\dagger$ to contain t_0, rather than to straddle it.

Again, this leads to a more stringent condition. In this case, it provides a solution to instances such as $x_{\mathrm{SPIKE}}[t]$ of Example 4.2, where there is an isolated discontinuous value at $t_0 = 0$. (An equivalent solution might be to impose a continuity condition.)

5.5.2 Flow Velocity and Completeness

It may seem surprising that the proof Lemma 5.7 appealed to completeness (it surprised me). Does completeness really provide a bridge between our different but related notions of velocity through a given instant and velocity on an interval? The following example shows how Lemma 5.7 can fail if the indexing intervals involved are subsets of the rational numbers, rather than real.

Example 5.7. Put $K = \{t \in \mathbb{Q} \mid 0 \leq t \leq 3\}$ and $I = \{t \in \mathbb{Q} \mid 1 \leq t \leq 2\}$. The definition of a fluent in Definition 3.1 requires the indexing interval to be real, but one can readily check that the concepts involved in Lemma 5.7 make sense for rational intervals and parameter values as well.

Now define a rationally indexed fluent z as follows:

$$z[t] = \begin{cases} t+1 & \text{if } t^2 < 2; \\ t-1 & \text{otherwise.} \end{cases}$$

for all $t \in K$.

When $t_0 \in I$ and $t_0^2 < 2$, we can always find $\varepsilon > 0$ such that $t^2 < 2$ for each $t \in I^\dagger = [t_0 - \varepsilon, t_0 + \varepsilon] \subseteq K$; and hence $z|_{I^\dagger}[t] = t + 1$ and $z \gtrdot 0$ on $I^\dagger$ through t_0.

When $t_0 \in I$ and $t_0^2 \geq 2$, it must be the case that $t_0^2 > 2$ because t_0 is necessarily rational. Hence, we can produce a similar argument to show that for this t_0 we also have $z \gtrdot 0$ through t_0.

But it isn't the case that $z \gtrdot 0$ on I, since, for example, $z[\min I] = 2 > z[\max I] = 1$: the conclusion of Lemma 5.7 no longer applies.

This example motivates the following theorem and its proof. Put informally, it says that we can show that the real numbers are complete from the following general assertion about fluents: if any fluent z that has positive flow velocity through each instant in an indexing interval, then it must have positive flow velocity on that interval as a whole.

For convenience, we appeal to our definition of completeness along an interval, as described in Section 2.3.

Theorem 5.10. *Assume the following proposition:*

- *Given any fluent z defined on an indexing interval K and any non-trivial closed sub-interval $I \subseteq K^o$ for which $z \gtrdot 0$ through each $t_0 \in I$, then z satisfies $z \gtrdot 0$ on I.*

Then:

- *Every bounded lower property on a finite closed interval has a supremum.*

Proof. We sketch a proof. Given a bounded lower property P on the finite closed interval $K = [T_1, T_2]$, our aim is to exhibit a supremum $c = \sup P \in K$.

Take each of the following three possibilities for P in turn.

- For any $t \in K$, $P(t)$ only holds when $t = T_1$. In that case, just set $c = T_1$, which satisfies the conditions for $\sup P$.
- If $P(t)$ holds for every $t \in K$ with $t < T_2$, then we can set $c = T_2$.
- Because P is a bounded lower property, the only remaining possibility is that we can find $t_1 > T_1$ and $t_2 < T_2$ with $P(t_1)$ true and $P(t_2)$ false. Then P is a bounded lower property on the sub-interval $I = [t_1, t_2] \subseteq K^o$.

  We assume there is no supremum for P in I and show it leads to a contradiction.

  Put $L = \Delta I > 0$, and define a fluent z on K as follows:

  $$z[t] = \begin{cases} t + L & \text{if } P(t) \text{ holds;} \\ t - L & \text{otherwise.} \end{cases}$$

  A similar argument to Example 5.7, shows that when $P(t_0)$ is true we can find a sub-interval $I^\dagger$ straddling t_0, where $z|_{I^\dagger}[t] = t + L$ and $z \gtrdot 0$ on $I^\dagger$ through t_0; and when $P(t_0)$ is false we also have $z \gtrdot 0$ through t_0.

  Hence, for all $t \in I$, $z \gtrdot 0$ through t_0; and, by the hypothesis of the theorem, this implies $z \gtrdot 0$ on I. But $z[T_1] = T_1 + L = T_2 > T_1 = T_2 - L = z[T_2]$, which is a contradiction ※. There must therefore be a supremum $c = \sup P \in I$ in this case, too; and this must act as a supremum for P applied to any $t \in K$ (not just for P restricted to I).

□

Chapter 6

Working with Flow Velocities

6.1 Bounded Flow Velocity

The concept of bounded flow velocity is a powerful one that seems natural for fluents, but plays little role in classical analysis. Fluents with bounded flow velocity have relatively rich properties that are obvious forerunners of those in differential calculus.

Definition 6.1. A fluent x has *bounded flow velocity through* t_0 iff x flows faster than velocity u and slower than velocity v through t_0 for some $u, v \in \mathbb{R}$ (ie. for some $u, v \in \mathbb{R}$, $u\psi_1 \lessdot x \lessdot v\psi_1$).

We call any such u a *strict lower flow velocity bound through* t_0 for the fluent x; and dually, any such v a *strict upper flow velocity bound through* t_0.

Remark 6.1. Most of the definitions and their consequences in this chapter will naturally apply to both fluents and fluent continuations, because they arise from flow velocities through a given time-like parameter instant. They can be defined for any time-like instant t_0 if a suitable finite non-trivial closed sub-interval exists for the fluent or fluent continuation, provided it straddles t_0.

Remark 6.2. If a fluent's flow velocity is faster than u and slower than v through t_0, then the respective restriction intervals $I^\dagger$ and $I^\ddagger$ in Definition 5.5 may not be equal. In that case, we can construct a common indexing interval I^* straddling t_0 by setting $I^* = I^\dagger \cap I^\ddagger$; and then, for every sub-interval $J \subseteq I^*$ containing t_0, we will have $u\,\Delta J < \Delta x|_J < v\,\Delta J$.

The following consequence of this definition is useful in visualisation; for example, when comparing flow velocity with classical differentiation in Section 6.10.

Corollary 6.1. *Suppose the fluent x with indexing interval I has bounded flow velocity through $t_0 \in I^o$ and $u\psi_1 \lessdot x \lessdot v\psi_1$ through t_0 for $u, v \in \mathbb{R}$. For any*

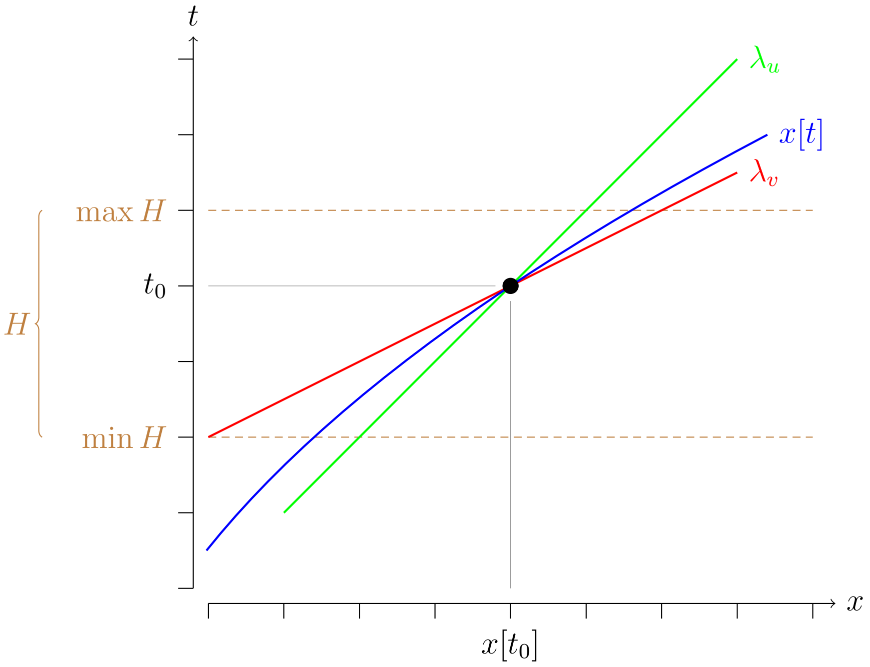

Figure 6.1: When $u\psi_1 \lessdot x \lessdot v\psi_1$ through t_0, for some finite closed sub-interval H straddling t_0 the fluent $x|_H$ lies within the skew cone defined by the linear fluents $\lambda_u[t] = u\,(t - t_0) + x[t_0]$ and $\lambda_v[t] = v\,(t - t_0) + x[t_0]$ for $t \in H$.

velocity-like constant $w \in \mathbb{R}$*, define the linear fluent continuation* $\lambda_w[t] = w\,(t - t_0) + x[t_0]$ *(we'll apply this to* u *and* v*).*

Then there is a finite closed sub-interval $H \subseteq I$ *straddling* t_0 *such that, for all* $t \in H$*, the linear fluents* $\lambda_u[t] = u\,(t - t_0) + x[t_0]$ *and* $\lambda_v[t] = v\,(t - t_0) + x[t_0]$ *bracket* $x[t]$ *in the following sense:*

- *if* $t < t_0$*, then* $\lambda_v[t] < x[t] < \lambda_u[t]$*;*
- *if* $t > t_0$*, then* $\lambda_u[t] < x[t] < \lambda_v[t]$*;*
- *and, trivially, if* $t = t_0$*, then* $\lambda_v[t] = x[t] = \lambda_u[t] = x[t_0]$*.*

The converse also applies. When the three bulleted conditions hold for $u, v \in \mathbb{R}$*,* $t \in H \subseteq I$ *and* $t_0 \in H^o$*, then* $u\psi_1 \lessdot x \lessdot v\psi_1$ *through* t_0 *and* x *has bounded flow velocity through* t_0*.*

Proof. We prove the original implication first and then its converse.

1. *Assuming* x *has bounded flow velocity through* t_0: Suppose $u\psi_1 \lessdot x \lessdot v\psi_1$ through t_0. This is equivalent to $\lambda_u \lessdot x \lessdot \lambda_v$, noting that $\lambda_w = w\psi_1 - wt_0 + x[t_0]$ (and so $\Delta\lambda_w|_J = w\,\Delta J = w\,\Delta\psi_1|_J$ for any candidate test interval J).

There is therefore a finite closed sub-interval $H \subseteq I$ straddling t_0 for which $\Delta\lambda_u|_J < \Delta x|_J < \Delta\lambda_v|_J$, for every non-trivial closed sub-interval $J \subseteq H$ containing t_0.

The result follows by separately considering intervals of form $J = [t, t_0]$ (where $t < t_0$) and $J = [t_0, t]$ (where $t > t_0$) for $t \in H$. In either case, the relation $\Delta\lambda_u|_J < \Delta x|_J < \Delta\lambda_v|_J$ expands to the corresponding inequality in $x[t]$.

The last relation for $t = t_0$ is trivally true by our definition of λ_u and λ_v.

2. *Assuming $x[t]$ satisfies the three bulleted conditions*: We show that $u\psi_1 \lessdot x \lessdot v\psi_1$ through t_0 and hence x is bounded through t_0.

   Choose any non-trivial closed sub-interval $J = [t_1, t_2] \subseteq H$ containing t_0, so $t_1 < t_2$ and $t_1 \leq t_0 \leq t_2$. We wish to show that $u\,\Delta J < \Delta x|_J < v\,\Delta J$.

   But this follows if we consider the separate intervals $J_1 = [t_1, t_0] \subseteq H$ and $J_2 = [t_0, t_2] \subseteq H$, noting that at least one of them is non-trivial and $\Delta J = \Delta J_1 + \Delta J_2$. We also have $\Delta x|_J = (x[t_0] - x[t_1]) + (x[t_2] - x[t_0])$.

   This is true for any such J, so that $u\psi_1 \lessdot x \lessdot v\psi_1$ through t_0 as required.

This completes the proof. □

Taken together, the three bulleted conditions say that the fluent plot of x lies within the skew cone defined by λ_u and λ_v, where we consider the points $(x[t], t)$ for all $t \in H$, including the vertex at $(x[t_0], t_0)$. (This is illustrated in Figure 8.2.)

6.2 Determining Flow Velocity Bounds

We now explore what happens when we try to tighten the bounds u and v in our definition. Suppose that the fluent x has bounded flow velocity through t_0, and so that for some $u, v \in \mathbb{R}$ we have $u\psi_1 \lessdot x \lessdot v\psi_1$. Then necessarily $u\psi_1 \lessdot v\psi_1$ and $u < v$ from the results of previous chapter. Hence, the real interval $[u, v]$ is well-defined, non-trivial, finite and closed.

For $w \in [u, v]$, define $L(w)$ to be the property $w\psi_1 \lessdot x$ through t_0. Then $L(u)$ holds; but not $L(v)$, because $x \lessdot v\psi_1$ through t_0; and if $L(w)$ holds and $w^\dagger < w$, then $L(w^\dagger)$ holds. In other words, L is a bounded lower property on interval $[u, v]$, so from Theorem 2.6 it has a supremum, which we write as $\grave{x} = \sup L$.

By definition of the supremum for L, if $u \leq w < \grave{x}$ then $w\psi_1 \lessdot x$; and by transitivity this relation also holds when $w < u$, ie. for all $w < \grave{x}$. The relation fails for all $w > \grave{x}$.

Definition 6.2. When fluent x has bounded flow velocity, we call the value $\grave{x}$ (pronounced "x-grave") in the previous construction the *greatest lower flow velocity of x through t_0*. We also denote this 'glfv x'; or occasionally $\grave{x}|_{t_0}$ if we wish to make the time-like instant t_0 explicit in the notation.

Dually, we can construct a value $\acute{x}$ (pronounced "x-acute") called the *least upper flow velocity of x through t_0*, which is the infimum of the bounded upper property $U(w) \equiv (x \lessdot w\psi_1$ through $t_0)$. We also denote this as 'lufv x'; and occasionally $\acute{x}|_{t_0}$.

The terminology was chosen to echo the names *greatest lower bound* and *least upper bound* in classical analysis, where they are synonyms for the *supremum* and *infimum* of a bounded set of real numbers. It's a little problematic, because in many cases $\grave{x}\psi_1$ and $\acute{x}\psi_1$ may not be strict lower or upper flow velocity bounds for x through t_0. We can only guarantee that $u\psi_1 \lessdot x \lessdot v\psi_1$ through t_0 when $u < \grave{x}$ and $\acute{x} < v$; and the relations may or may not hold when $u = \grave{x}$ and $\acute{x} = v$.

For those who prefer the more orthodox methods of modern analysis, an equivalent set-based definition would be the following.

Definition (Set-based equivalent of Definition 6.2)**.** When fluent x has bounded flow velocity through t_0, we can define two real values:

$$\grave{x} = \sup\{w \in \mathbb{R} \mid w\psi_1 \lessdot x \text{ through } t_0\};$$

and:

$$\acute{x} = \inf\{w \in \mathbb{R} \mid x \lessdot w\psi_1 \text{ through } t_0\}.$$

Here $\grave{x}$ is the *greatest lower flow velocity of x through t_0* (also written glfv x through t_0); and $\acute{x}$ is the *least upper flow velocity of x through t_0* (also written lufv x through t_0).

We may also write $\grave{x}|_{t_0}$ and $\acute{x}|_{t_0}$ if we wish to make t_0 explicit notationally.

Remark 6.3. The condition that x has bounded flow velocity through t_0 ensures that the two sets involved in this set-based definition are non-empty and hence their respective supremum and infimum both exist.

In our original construction and definition, we've established the bulk of the following fundamental theorem for a fluent with bounded flow velocity.

Theorem 6.2. *Every fluent x with bounded flow velocity through t_0 has a well-defined greatest lower flow velocity $\grave{x}$ through t_0, where:*

- $w < \grave{x} \Rightarrow (w\psi_1 \lessdot x$ *through* $t_0)$*; and*
- $w > \grave{x} \Rightarrow \neg(w\psi_1 \lessdot x$ *through* $t_0)$*;*

  or equivalently: $(w\psi_1 \lessdot x$ *through* $t_0) \Rightarrow w \leq \grave{x}$*.*

Dually, it has a least upper flow velocity $\acute{x}$ *through* t_0*, where:*

- $w > \acute{x} \Rightarrow (w\psi_1 \gtrdot x$ *through* $t_0)$*; and*
- $w < \acute{x} \Rightarrow \neg(w\psi_1 \gtrdot x$ *through* $t_0)$*;*

  or equivalently: $(w\psi_1 \gtrdot x$ *through* $t_0) \Rightarrow w \geq \acute{x}$*.*

In all cases, $\grave{x} \leq \acute{x}$*.*

Proof. The only remaining assertion to be proved is that $\grave{x} \leq \acute{x}$ through t_0.

Suppose instead $\grave{x} > \acute{x}$. Then $w = (\grave{x} + \acute{x})/2$ satisfies $w > \acute{x}$; and hence $w\psi_1 \gtrdot x$ through t_0. But also $w < \grave{x}$; and hence $w\psi_1 \lessdot x$ through t_0. But these can't both be true ※.

Hence, we must have $\grave{x} \leq \acute{x}$. ☐

This leads to the following corollary, which can be useful for determining the greatest lower and least upper flow velocity of a given fluent.

Corollary 6.3. *Suppose fluent* x *has bounded flow velocity through* t_0*, and* $c \in \mathbb{R}$*.*

1. *If for all* $u \in \mathbb{R}$ *with* $u < c$*, we have* $u\psi_1 \lessdot x$ *through* t_0*; then* $\grave{x} \geq c$*.*
2. *If for all* $v \in \mathbb{R}$ *with* $c < v$*, we have* $x \lessdot v\psi_1$ *through* t_0*; then* $\acute{x} \leq c$*.*
3. *If for all* $u, v \in \mathbb{R}$ *with* $u < c < v$*, we have* $u\psi_1 \lessdot x \lessdot v\psi_1$ *through* t_0*; then* $\grave{x} = \acute{x} = c$*.*

Proof. We take each assertion in order.

1. To prove the first assertion, we assume $\grave{x} < c$ and show it leads to a contradiction. Put $u = \frac{1}{2}(\grave{x} + c)$. Then $\grave{x} < u$, which implies $\neg(u\psi_1 \lessdot x)$ through t_0.

   But $u < c$, which by hypothesis implies $u\psi_1 \lessdot x$ ※.

   Hence, we must have $\grave{x} \geq c$, as stated.
2. The second assertion is proved dually by setting $v = \frac{1}{2}(c + \acute{x})$.
3. We prove the final assertion by combining the other two. By our previous results, we can deduce that $\acute{x} \leq c \leq \grave{x}$.

   But we also know that in general $\grave{x} \leq \acute{x}$.

   Hence, $\grave{x} = \acute{x} = c$ through t_0.

☐

When $\grave{x} = \acute{x} = c$, as in the last case of this corollary, then we can use Newton's dot notation for the common value $\dot{x} = c$ and say that x has a unique flow velocity limit through t_0. This situation is explored more fully in the next section.

Example 6.1. We can apply Corollary 6.3 to the fluent continuation ψ_1 to get the reassuring result that for all $t_0 \in \mathbb{R}$:

$$\grave{\psi}_1 = \acute{\psi}_1 = 1 \text{ through } t_0.$$

Hence, ψ_1 has the unique flow velocity limit $\dot{\psi}_1 = 1$ through t_0.

We next look at a case where $\grave{x} < \acute{x}$ through a particular t_0.

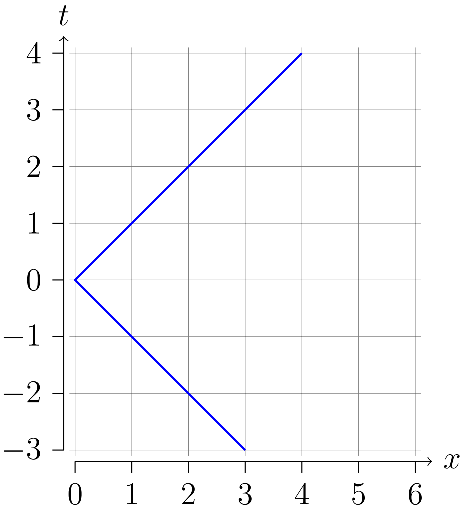

Figure 6.2: The fluent continuation $x[t] = |t|$ from Example 6.2.

Example 6.2. We return to the fluent continuation $x[t] = |t|$ and choose $t_0 \in \mathbb{R}$. We noted in an earlier example that $u\psi_1 \lessdot x \lessdot v\psi_1$ through t_0 for all $u < -1$ and for all $v > +1$. Hence x is bounded flow velocity; and $\grave{x} \geq -1$, while $\acute{x} \leq +1$ through t_0. In summary, for any t_0, we have $-1 \leq \grave{x} \leq \acute{x} \leq +1$ through t_0.

In fact, x is indistinguishable from ψ_1 in indexing intervals of the form $[0, T_2]$ where $T_2 > 0$, so $\grave{x} = \acute{x} = +1$ through any instant $t_0 > 0$. We might also (correctly) anticipate that $\grave{x} = \acute{x} = -1$ through any instant $t_0 < 0$.

What about when $t_0 = 0$? Consider an interval $I^\dagger = [T_1^\dagger, T_2^\dagger]$ straddling $t_0 = 0$. We can always find a non-trivial closed sub-interval $J = [t_1, t_2] \subseteq I^\dagger$, which fails the requirement $\Delta x|_J < -1.\Delta J$; eg. $J = [T_1, 0]$ has $\Delta x|_J = -1.\Delta J$. This failure implies $\grave{x} \leq -1$ through $t_0 = 0$; and it takes only a few more steps to conclude that $-1 = \grave{x} < \acute{x} = +1$ through $t_0 = 0$.

6.3 Unique Flow Velocity Limit

This section contains several more examples of flow velocity bounds.

In the following examples, we'll find that the two limiting values $\grave{x} \leq \acute{x}$ are equal through each instant t_0 straddled by a suitable indexing interval. As one might already expect, this is quite common when there is an explicit mathematical relation between x and its parameter t. When there is an indexing interval I in which every $t_0 \in I$ satisfies $\grave{x} = \acute{x}$ through t_0, then we can follow Newton in deriving a new fluent $\dot{x}[t] = (\grave{x}$ through $t) = (\acute{x}$ through $t)$, which we call the *fluxion* of x. For the moment, we define some terminology for the local case through a specific t_0.

Fluxions are examined in Chapter 7. In our account, they're defined with respect to a particular parametrisation. In many cases, their basic formal properties derive directly from those for fluents with bounded flow velocity, which is one motive for continuing to explore these here. Another is that the class of fluents with bounded flow velocity is larger than those from which a fluxion can be derived, as indicated by the familiar case of $x[t] = |t|$ above. We'll wait until the relevant chapter before defining the fluxion as a derived fluent, but in the meantime will borrow the dot notation by writing $\dot{x}$ or $\dot{x}|_{t_0}$ when we know $\grave{x} = \acute{x}$ through a given t_0.

When $\dot{x}|_{t_0}$ applies to a specific t_0, we might be tempted to talk informally of the "fluxion being defined at t_0". The following definition uses the more precise phrase *unique flow velocity limit through* t_0. It makes no claim to the limits being defined uniquely for all t along a non-trivial indexing interval, which would need to be the case for a fully constructed fluxion.

Definition 6.3. Suppose the fluent x has bounded flow velocity through t_0 and that $\grave{x} = \acute{x}$ through t_0. Then we say that x has a *unique flow velocity limit through* t_0. We call the common value the *(unique) flow velocity limit through* t_0 and denote it by $\dot{x} = \grave{x} = \acute{x}$. We may also write this $\dot{x}|_{t_0}$, when making t_0 explicit.

Remark 6.4. One can show that a fluent x has a unique flow velocity limit through t_0 iff the evaluation function $\nu\colon t \mapsto x[t]$ is classically differentiable at t_0.

Section 6.10 compares the two approaches in more detail.

Remark 6.5. The dot notation for fluents contains a potential pitfall if you're used to using it in physical models to denote a time derivative. It only makes sense for a given fluent if the parametrisation is understood (as is the case for $\grave{x}$ and $\acute{x}$ through t_0), and is then with respect to that parameter.

Physical time may itself be regarded as a fluent. If we have chosen to denote *physical* time by the variable t, it may then itself be parametrised to form a fluent $t[s]$, where we're using s to denote our chosen parameter.

In that case, when $t[s]$ has bounded flow velocity (with respect to s through some appropriate s_0), then we may find that $\grave{t} = \acute{t} \neq 1$. Regarding t as a fluent, we would then have $\dot{t} \neq 1$, which may be somewhat surprising to a physicist who equates $\dot{x}$ with $\frac{\partial x}{\partial t}$ for any variable x.

Next, we provide some examples of fluent continuations with a unique flow velocity limit at each point.

Example 6.3. Consider the fluent continuation $x[t] = t^2$. Then if $J = [t_1, t_2]$ we know from Example 5.4 that $\Delta x|_J = (t_1 + t_2)\,\Delta J$.

Now fix the point t_0 and set the velocity value $c = 2t_0$. Choose any $u < c$. We wish to show $u\psi_1 \lessdot x$ on some $I^\dagger$ through t_0, which for the moment we can only do by first principles. This means that we want to find an interval $I^\dagger = [T_1^\dagger, T_2^\dagger]$ straddling t_0, where for each closed non-trivial interval $J \subseteq I^\dagger$ containing t_0 we have $u\,\Delta J < \Delta x|_J$. To achieve this, we need to constrain t_1 (and hence $T_1^\dagger$) against being too small.

Consider the non-trivial interval $J = [T_1^\dagger, t_0]$. To ensure $u\,\Delta J < \Delta x|_J$, we require $T_1^\dagger + t_0 > u$ for some $T_1^\dagger < t_0$; so $T_1^\dagger = t_0 - \tau$ for some $\tau > 0$. Substituting for $T_1^\dagger$, we now require $t_0 - \tau + t_0 > u$, ie. $c - \tau > u$, which we can achieve by setting $\tau = \frac{1}{2}(c - u) > 0$.

If $J = [t_0, t_2]$, then $\Delta x|_J > u\,\Delta J$ for any t_2, so we can choose any $T_2^\dagger > t_0$, eg. $T_2^\dagger = t_0 + 1$.

We therefore consider $I^\dagger = [T_1^\dagger, T_2^\dagger]$, where $T_1^\dagger = t_0 - \frac{1}{2}(c - u)$ and $T_2^\dagger > t_0 + 1$.

Now for any $J \subseteq I^\dagger$ with $t_0 \in J$, we have:

$$\begin{aligned}\Delta x|_J &= (t_1 + t_2)\,\Delta J\\ &\geq (T_1^\dagger + t_0)\,\Delta J\\ &> u\,\Delta J.\end{aligned}$$

We've therefore shown that for any $u < c = 2t_0$ we can construct a non-trivial closed interval $I^\dagger$ containing t_0 such that $u\psi_1 \lessdot x$ on $I^\dagger$ through t_0. Hence $\grave{x} \geq c$.

A dual argument shows that for any $v > c$ we must have $v\psi_1 \gtrdot x$ through t_0; hence $\acute{x} \leq c$.

Because these values of u and v are arbitrary, we can apply Corollary 6.3 to show that for this fluent, $\acute{x} = \grave{x} = \dot{x} = c = 2t_0$ through t_0, ie. it has a unique flow velocity limit.

Can we say that $c\psi_1 \lessdot x$ through t_0? We show the answer is no.

If so, we could find $I^\ddagger = [T_1^\ddagger, T_2^\ddagger]$ straddling t_0, where each qualifying $J \subseteq I^\ddagger$ satisfies $\Delta x|_J > c\,\Delta J$. We show that finding such an $I^\ddagger$ is impossible by considering $J = [T_1^\ddagger, t_0]$.

This J is a finite non-trivial closed sub-interval of $I^{\ddagger}$ containing t_0, so it qualifies; but in this case:

$$\begin{aligned}\Delta x|_J &= (T_1^{\ddagger} + t_0)\,\Delta J\\ &< (t_0 + t_0)\,\Delta J\\ &= c\,\Delta J.\end{aligned}$$

Hence $\neg(c\psi_1 \lessdot x \text{ through } t_0)$.

The previous examples show that it isn't the general case that $\grave{x}\,\psi_1 \lessdot x$ through a time-like instant t_0, which may seem counter-intuitive (or to disqualify the term greatest lower flow velocity). It follows from the fact that when $\grave{x} = \sup L$, it depends on the choice of x and t_0 whether the predicate expression $L(\grave{x})$ will be true or false. (Similarly, the supremum or greatest lower bound of a non-empty set may or may not lie in that set.) There is, of course, a dual result for $\acute{x}$.

Perhaps even more surprisingly, there are cases where we have $\acute{x}\,\psi_1 \lessdot x$ through t_0, despite $\acute{x}$ being a least upper flow velocity (and dual cases where $x \lessdot \grave{x}\,\psi_1$). The fluent $x[t] = t^3$ provides such an example (through $t_0 = 0$).

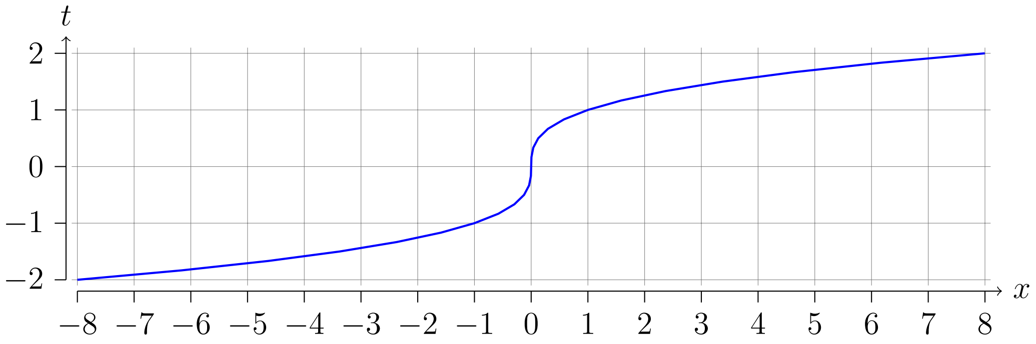

Figure 6.3: The fluent continuation $x[t] = t^3$ from Example 6.4.

Example 6.4. Now consider the fluent continuation $x[t] = t^3$ and choose any $t_0 \in \mathbb{R}$. For any indexing interval $I = [T_1, T_2]$ straddling t_0 and any finite non-trivial closed sub-interval $J = [t_1, t_2] \subseteq I$ containing t_0, we have:

$$\begin{aligned}\Delta x|_J &= (t_1^2 + t_1 t_2 + t_2^2)(t_2 - t_1)\\ &= \tfrac{1}{2}\left(t_1^2 + t_2^2 + (t_1 + t_2)^2\right)\,\Delta J\\ &> 0\end{aligned}$$

Hence $x \gtrdot 0$ on I; ie. x has positive flow velocity on I; so it has positive flow velocity through $t_0 \in I^o$; and this true for all $t_0 \in \mathbb{R}$.

Now consider $t_0 = 0$. As elsewhere, we must have $0 \leq \grave{x} \leq \acute{x}$ through $t_0 = 0$. We now prove that $\acute{x} = 0$ through $t_0 = 0$; and hence $\grave{x} = \acute{x} = \dot{x} = 0$ there.

Choose any $v > 0$; and form the interval $I^\dagger = [T_1^\dagger, T_2^\dagger]$, where $T_1^\dagger = -\frac{1}{4}\sqrt{v} = -T_2^\dagger$, which straddles $t_0 = 0$. Then $I^\dagger$ straddles $t_0 = 0$; and for any finite non-trivial closed sub-interval $J \subseteq I^\dagger$ containing $t_0 = 0$, the above equations show that $\Delta x|_J < v\psi_1$.

Thus $x \lessdot v\psi_1$ through $t_0 = 0$. Since this is true of every $v > 0$, we must have $\acute{x} \leq 0$ and the rest follows.

Note that this fluent has the perhaps surprising property that $\acute{x}\,\psi_1 \lessdot x$ through $t_0 = 0$.

In Table 6.1, we summarise the main results for each of our examples so far.

x	$\grave{x}$ through t_0	$\acute{x}$ through t_0
1	0	0
t	1	1
t^2	$2t_0$	$2t_0$
t^3	$3t_0^2$	$3t_0^2$
$\|t\|$	$\begin{cases} -1 & \text{if } t_0 < 0 \\ -1 & \text{if } t_0 = 0 \\ +1 & \text{if } t_0 > 0 \end{cases}$	$\begin{cases} -1 & \text{if } t_0 < 0 \\ +1 & \text{if } t_0 = 0 \\ +1 & \text{if } t_0 > 0 \end{cases}$

Table 6.1: Some examples of greatest lower and least upper flow velocities.

6.4 Basic Properties

A number of basic properties are useful for calculating the greatest lower flow velocity and least upper flow velocity.

Theorem 6.4. *Suppose $a, b \in \mathbb{R}$; and that x, y, z are any three real-valued fluents defined on a common indexing interval I, which straddles $t_0 \in I^o$. Assume also that x and y have bounded flow velocity through t_0. Then the following basic relations hold for the flow velocity through t_0.*

1. *(Bracketing) If $x \lessdot z \lessdot y$, then z has bounded flow velocity and $\grave{x} \leq \grave{z} \leq \acute{z} \leq \acute{y}$.*
2. *(Duality) If fluent $w = \operatorname{opp} x$ (its opposite$^{(Def\ 3.3)}$), then w has bounded flow velocity through $-t_0$ and $\grave{w} = -\acute{x} \leq -\grave{x} = \acute{w}$ through t_0 for x and $-t_0$ for w.*
3. *(Linear relabelling) More generally, define a relabelling $r \colon H \to I$, where $s \mapsto as + b$ for each $s \in H$ and $t_0 = r(s_0)$ for some $s_0 \in H$. This induces a fluent $w = x \circ r$, for which $w[s] = x[t]$, where $t = r(s)$.*

   If $a \geq 0$, then w has bounded flow velocity through s_0 and $\grave{w} = a\grave{x} \leq a\acute{x} = \acute{w}$ through t_0 for x and s_0 for w.
4. *(Translational invariance) If $z = x + a$, then z has bounded flow velocity and $\grave{x} = \grave{z} \leq \acute{z} = \acute{x}$.*
5. *(Non-negative scaling) If $0 \leq a$ and $z = ax$, then z has bounded flow velocity and $a\grave{x} = \grave{z} \leq \acute{z} = a\acute{x}$.*
6. *(Negative scaling) If $z = -x$, then z has bounded flow velocity and $-\acute{x} = \grave{z} \leq \acute{z} = -\grave{x}$.*
7. *(Summation 1) If $z = x + y$, then z has bounded flow velocity and $\grave{x} + \grave{y} \leq \grave{z} \leq \acute{z} \leq \acute{x} + \acute{y}$.*

   Also, $\grave{z} \leq \grave{x} + \acute{y} \leq \acute{z}$; and $\grave{z} \leq \acute{x} + \grave{y} \leq \acute{z}$.
8. *(Summation 2) Hence, if $z = x + y$ and $\grave{y} = \acute{y} = \dot{y}$, then z has bounded flow velocity and $\grave{x} + \dot{y} = \grave{z} \leq \acute{z} = \acute{x} + \dot{y}$.*

Proof. Each assertion follows from the definition of greatest lower flow velocity and least upper flow velocity. We prove the case where $z = x + y$ as an illustration.

- *To show* $\grave{x} + \grave{y} \leq \grave{z}$: Put $c = \grave{x} + \grave{y}$ and choose a test velocity $w < c$; so that $\varepsilon = c - w > 0$.

Now put $u = \grave{x} - \frac{1}{2}\varepsilon < \grave{x}$, so that $u\psi_1 \lessdot x$; and similarly $v = \grave{y} - \frac{1}{2}\varepsilon < \grave{y}$, so $v\psi_1 \lessdot y$.

This implies (from Lemma 5.9) that $w\psi_1 = (u{+}v)\psi_1 = u\psi_1{+}v\psi_1 \lessdot x{+}y = z$. Hence $w \leq \grave{z}$; and this is true for any $w < c$; so it must also be true that $c \leq \grave{z}$.

- *To show $\acute{z} \leq \acute{x} + \acute{y}$:* We can prove the dual relation $\acute{z} \leq \acute{x} + \acute{y}$ similarly; or by appealing to the result that $\acute{z} = -\operatorname{glfv}(\operatorname{opp} z)$ and expanding in terms of x and y:

  $$\begin{aligned} -\acute{z} &= \operatorname{glfv}(\operatorname{opp} z) \\ &= \operatorname{glfv}(\operatorname{opp} x + \operatorname{opp} y) \\ &= \operatorname{glfv}(\operatorname{opp} x + \operatorname{opp} y) \\ &\geq \operatorname{glfv}(\operatorname{opp} x) + \operatorname{glfv}(\operatorname{opp} y) \\ &= -(\acute{x} + \acute{y}) \end{aligned}$$

  Hence the required result that $\acute{z} \leq \acute{x} + \acute{y}$.

- *To show $\grave{z} \leq \grave{x} + \acute{y}$:* Choose any $w > \grave{x} + \acute{y}$ and put $\varepsilon = w - (\grave{x} + \acute{y}) > 0$. We aim to prove $\neg(w\psi_1 \lessdot z)$ through t_0 by showing that, for any closed sub-interval $I^\dagger \subseteq I$ straddling t_0, we can construct a closed sub-interval $J \subseteq I^\dagger$ containing t_0 such that $\Delta z|_J \leq w\,\Delta J$.

  Write $u = \grave{x} + \frac{1}{2}\varepsilon$ and $v = \acute{y} + \frac{1}{2}\varepsilon$, so $w = u + v$ where $u > \grave{x}$ and $v > \acute{y}$. This implies $\neg(u\psi_1 \lessdot x)$ and $y \lessdot v\psi_1$ through t_0.

  We can find a closed sub-interval $I^\ddagger \subseteq I$ straddling t_0 where $y \lessdot v\psi_1$ on $I^\ddagger$ through t_0. Now choose any closed sub-interval $I^\dagger \subseteq I$ straddling t_0; and put $I^* = I^\dagger \cap I^\ddagger$, which also straddles t_0.

  Because $\neg(u\psi_1 \lessdot x)$ through t_0, there's a closed sub-interval $J \subseteq I^* \subseteq I^\dagger$, where $\Delta x|_J \leq u\,\Delta J$.

  And because $y \lessdot v\psi_1$ on $I^\ddagger$ through t_0 and $J \subseteq I^\ddagger$, then $\Delta y|_J < v\,\Delta J$.

  Hence $\Delta z|_J = \Delta x|_J + \Delta y|_J < (u + v)\,\Delta J = w\,\Delta J$; so J has the required properties; and we infer that $\neg(w\psi_1 \lessdot z)$ through t_0.

  Hence $\grave{z} \leq w$ for each $w > \grave{x} + \acute{y}$. Therefore $\grave{z} \leq \grave{x} + \acute{y}$.

  By symmetry, $\grave{z} \leq \acute{x} + \grave{y}$. And dually, using opposite fluents, we must have $\grave{x} + \acute{y} \leq \acute{z}$ and $\acute{x} + \grave{y} \leq \acute{z}$.

This completes the proof. $\square$

Example 6.5. We can apply these results as follows to find the flow velocity bounds of the constant fluent continuation $x[t] = a$, for a fixed $a \in \mathbb{R}$.

Start with the fluent continuation ψ_1. It has bounded flow velocity through any $t_0 \in \mathbb{R}$; hence so does $0\,\psi_1$; hence also $x[t] = 0\,\psi_1 + a$.

Therefore $\grave{x} = \text{glfv}(0\,\psi_1 + a) = 0\,\grave{\psi}_1 = 0$; and similarly $\acute{x} = 0$.

The presence of an inequality in the case of a sum of two fluents may seem surprising, especially when considering cases such as $\grave{x} + \grave{x} = 2\grave{x} = \text{glfv}(2x) = \text{glfv}(x + x)$. We explore these inequalities slightly further.

Corollary 6.5. *If fluents x and y have bounded flow velocity through t_0 and the fluent $z = x + y$, then z satisfies:*

$$|(\acute{x} - \grave{x}) - (\acute{y} - \grave{y})| \le \acute{z} - \grave{z} \le (\acute{x} - \grave{x}) + (\acute{y} - \grave{y}).$$

Therefore, when $\grave{y} = \acute{y} = \dot{y}$, this simplifies to:

$$\acute{x} - \grave{x} = \acute{z} - \grave{z}.$$

Proof. The main inequality follows directly by manipulating the inequalities for $z = x + y$ in Theorem 6.4.

For the simplified result, note that $\acute{x} - \grave{x} > 0$; hence $|\acute{x} - \grave{x}| = \acute{x} - \grave{x}$. ☐

6.5 Flow Velocity Product Rule

The product rule plays a key role in differential calculus, going back to Leibniz or perhaps earlier. It is presaged here by a version for fluents with bounded flow velocity. The next lemma begins with a special case, in preparation for the more widely applicable theorem.

Lemma 6.6. *Suppose that fluents x and y share an indexing interval I and they both have bounded flow velocity through $t_0 \in I^o$. Suppose also that $x[t_0] = y[t_0] = 0$.*

Then the fluent $z = xy$ is well-defined on I, and also has bounded flow velocity through t_0. Moreover, it satisfies $\grave{z} = \acute{z} = \dot{z} = 0$ through t_0.

Proof. If $u\psi_1 \lessdot x \lessdot v\psi_1$ and $u'\psi_1 \lessdot y \lessdot v'\psi_1$ through t_0, then we can set $c = \max\{|u|, |v|, |u'|, |v'|\} > 0$; so that $-c\psi_1 \lessdot x \lessdot +c\psi_1$ and $-c\psi_1 \lessdot y \lessdot +c\psi_1$ through t_0.

Furthermore, we can construct a common non-trivial finite closed sub-interval $H \subseteq I$ straddling t_0, such that for every closed sub-interval $J \subseteq H$ containing t_0:

$$\begin{aligned} &-c\,\Delta J < \Delta x < c\,\Delta J \\ \text{and } &-c\,\Delta J < \Delta y < c\,\Delta J \end{aligned}$$

Now choose a test velocity $w > 0$. Set $H^* = H \cap [t_0 - \delta, t_0 + \delta]$, where $\delta = w/2c^2 > 0$; so, by construction, $H^* \subseteq H$ is a non-trivial closed interval straddling t_0.

Then, for any closed sub-interval $J = [t_1, t_2] \subseteq H^*$ containing t_0:

$$\begin{aligned}
\Delta z|_J &= y[t_1]\,\Delta x|_J + x[t_2]\,\Delta y|_J \\
&\leq \big|\, y[t_1]\,\Delta x|_J + x[t_2]\,\Delta y|_J \,\big| \\
&\leq \big|\, y[t_1]\,\Delta x|_J \,\big| \;+\; \big|\, x[t_2]\,\Delta y|_J \,\big| \\
&< 2\,(c\delta)\,(c\Delta J) \\
&< w\,\Delta J,
\end{aligned}$$

where we've used the relations $|x[t_2]| = |x[t_2] - x[t_0]| < c|t_2 - t_0| \leq c\,\delta$; $|y[t_1]| = |y[t_0] - y[t_1]| < c|t_0 - t_1| \leq c\,\delta$; $|\Delta x|_J| < c\,\Delta J$; and $|\Delta y|_J| < c\,\Delta J$.

Hence $z \lessdot w\psi_1$ through t_0; and this is true for all $w > 0$.

We can show similarly that $w\psi_1 \lessdot z$ through t_0 for all $w < 0$.

Hence z has bounded flow velocity through t_0; and $\grave{z} = \acute{z} = \dot{z} = 0$ through t_0, as required. ☐

Theorem 6.7. *Suppose that fluents x and y share an indexing interval I and they both have bounded flow velocity through $t_0 \in I^o$. Suppose also that $x_0 \geq 0$ and $y_0 \geq 0$, where $x_0 = x[t_0] \in \mathbb{R}$ and $y_0 = y[t_0] \in \mathbb{R}$.*

Then the fluent $z = xy$ is well-defined on I, and has bounded flow velocity through t_0. Moreover, it satisfies the relations:

$$y_0\,\grave{x} + x_0\,\grave{y} \;\leq\; \grave{z} \;\leq\; \acute{z} \;\leq\; y_0\,\acute{x} + x_0\,\acute{y} \quad \textit{through } t_0.$$

Proof. The proof proceeds fairly mechanically, once we notice that fluents $x - x_0$ and $y - y_0$ fulfil the premisses of Lemma 6.6.

We note that $z = (x - x_0)(y - y_0) + y_0\,x + x_0\,y - x_0 y_0$ and apply Theorem 6.4 to deduce that z has bounded flow velocity through t_0, with:

$$\begin{aligned}
\acute{z} &= \mathrm{lufv}\,\big((x - x_0)(y - y_0) + y_0\,x + x_0\,y - x_0 y_0\big) \\
&\leq 0 + \mathrm{lufv}\,\big(y_0\,\acute{x}\big) + \mathrm{lufv}\,\big(x_0\,\acute{y}\big) + 0 \\
&= y_0\,\acute{x} + x_0\,\acute{y}.
\end{aligned}$$

The inequality for $\grave{z}$ follows dually. ☐

Example 6.6. We can apply this product rule to the fluent continuation $x[t] = t^2$, which is the product fluent $x = \psi_1\,\psi_1$.

Then, $\acute{x} \leq (1\,t_0 + 1\,t_0) = 2\,t_0$ through any $t_0 \in \mathbb{R}$.

From the dual result, we see that $\grave{x} = \acute{x} = \dot{x} = 2\,t_0$, echoing our earlier calculation from first principles.

6.6 Flow Velocity Intervals

This section is mainly notational.

When a fluent has bounded flow velocity through a given instant, it can be convenient to combine the greatest lower and least upper flow velocity values to form the corresponding finite non-empty closed interval.

Definition 6.4. Suppose fluent $x[t]$ has bounded flow velocity through $t = t_0$. We call the finite non-empty closed interval $[\grave{x}, \acute{x}]$ the *flow velocity interval of x through t_0*, which we denote by $[\partial x \,|\, \partial t]\,\big|_{t_0}$ or (for example, when t_0 is understood) just $[\partial x \,|\, \partial t]$.

(This notation is meant to evoke both classical differentiation and the resulting closed interval.)

When $\grave{x} = \acute{x} = \dot{x}$, ie. x has a unique flow velocity limit, then the flow velocity interval becomes trivial and we may emphasise this by writing it as the equivalent singleton set $\{\dot{x}\}$.

Example 6.7. If $x[t] = c\,|t|$ for a fixed $c > 0$, then we can write:

$$[\partial x \,|\, \partial t] = [-c, +c] \quad \text{through } t = 0.$$

When a fluent x develops in proportion to the absolute value of t, it can be compared conceptually to a physical object with constant velocity (such as a billiard ball) that undergoes an elastic collision at $x = 0$ when also $t = 0$. Its displacement travels downwards towards zero then back up again as t goes from negative to positive, before and after it bounces back along its trajectory. One can then ask what velocity (if any) should be assigned through the instant of collision. Given that the total kinetic energy plus strain energy is conserved, the flow velocity interval provides a provocative reply.

Theorem 6.8. *Fluent x has bounded flow velocity through t_0 iff x has a well-defined flow velocity interval $[\partial x \,|\, \partial t]$ through t_0.*

If this is the case, then each of the following applies:

- *x has a unique flow velocity limit through t_0 iff $[\partial x \,|\, \partial t]$ is trivial;*
- $w < [\partial x \,|\, \partial t] \implies (w\psi_1 \lessdot x$ *through* $t_0)$;
- $w > \min\,[\partial x \,|\, \partial t] \implies \neg(w\psi_1 \lessdot x$ *through* $t_0)$;

  or, equivalently: $(w\psi_1 \lessdot x$ *through* $t_0) \implies w \le \min\,[\partial x \,|\, \partial t]$;
- $w > [\partial x \,|\, \partial t] \implies (w\psi_1 \gtrdot x$ *through* $t_0)$;

- *$w < \max[\partial x \,|\, \partial t] \implies \neg(w\psi_1 \gtrdot x$ through $t_0)$;*

  or, equivalently: $(w\psi_1 \gtrdot x$ through $t_0) \implies w \geq \max[\partial x \,|\, \partial t]$.

Proof. This is essentially a restatement of Theorem 6.2. □

We can use the flow velocity interval to more concisely express Theorem 6.4 and the product rule of Theorem 6.7. To do so, we make some notational extensions for manipulating closed intervals.

Definition 6.5. Let $[u, v]$, $[u_1, v_1]$ and $[u_2, v_2]$ all be finite closed intervals (possibly trivial); and $a, b \in \mathbb{R}$. Then we use the following notational conventions:

- Define $[u, v] + b \equiv [u + b, v + b]$.
- If $a \geq 0$, define $a\,[u, v] \equiv [au, av]$.
- If $a < 0$, define $a\,[u, v] \equiv [av, au]$.
- Define $[u_1, v_1] + [u_2, v_2] \equiv [u_1 + u_2, v_1 + v_2]$.
- Define $[u : v] \equiv [\min\{u, v\}, \max\{u, v\}]$

The idea of the notation $[u : v]$ is that it can be helpful when the ordering of u and v can't be predicted by the context. For example, it enables us to write $a\,[u, v] + b = [au + b : av + b]$, which allows equally for a being either positive or negative. It arises in this document because we've stipulated in Chapter 2 that an interval written $[u, v]$ must satisfy $u \leq v$.

Corollary 6.9. *Let $[u, v]$, $[u_1, v_1]$ and $[u_2, v_2]$ all be finite closed intervals (possibly trivial), so $u \leq v$, etc; and suppose $a, b \in \mathbb{R}$. Then:*

- *$a\,[u, v] + b = [au + b : av + b]$.*
- *$a\,([u, v] + b) = a\,[u, v] + ab$.*
- *$a\,([u_1, v_1] + [u_2, v_2]) = a\,[u_1, v_1] + a\,[u_2, v_2]$.*
- *$[u_1, v_1] \subseteq [u_2, v_2]$ iff $u_2 \leq u_1 \leq v_1 \leq v_2$.*
- *If $[u_1, v_1] \subseteq [u_2, v_2]$, then $a\,[u_1, v_1] + b \subseteq a\,[u_2, v_2] + b$.*

Proof. Each result follows directly from Definition 6.5. □

Theorem 6.10. *Suppose $a, b \in \mathbb{R}$; and that x, y, z are any three real-valued fluents defined on a common indexing interval I, which straddles $t_0 \in I^o$. Assume also that x and y have bounded flow velocity through t_0. Then the following flow velocity intervals exist through t_0 and satisfy the corresponding relations.*

1. *If* $x \lessdot z \lessdot y$*, then* $[\partial z \,|\, \partial t] \subseteq [\grave{x}, \acute{y}]$.
2. *If fluent* $w = \operatorname{opp} x$*, then* $[\partial w \,|\, \partial s] = -\,[\partial x \,|\, \partial t]$ *through* t_0 *for* x *and* $s_0 = -t_0$ *for* w*; where* $s = -t$*, so* $s \in \operatorname{opp} I = -I$.
3. *Define a relabelling* $r\colon H \to I$*, where* $s \mapsto as + b$ *for each* $s \in H$ *and* $t_0 = r(s_0)$ *for some* $s_0 \in H$*. Construct the fluent* $w = x \circ r$*, for which* $w[s] = x[t]$*, where* $t = r(s)$.

   Then $[\partial w \,|\, \partial s] = a\,[\partial x \,|\, \partial t]$ *through* t_0 *for* x *and* s_0 *for* w.
4. $[\partial(ax + b) \,|\, \partial t] = a\,[\partial x \,|\, \partial t]$.
5. $[\grave{x} + \acute{y} : \acute{x} + \grave{y}] \;\subseteq\; [\partial(x + y) \,|\, \partial t] \;\subseteq\; [\partial x \,|\, \partial t] + [\partial y \,|\, \partial t]$.
6. *If* $\grave{y} = \acute{y} = \dot{y}$*, then* $[\partial(x + y) \,|\, \partial t] \;=\; [\partial x \,|\, \partial t] + \dot{y}$.
7. $[\partial(xy) \,|\, \partial t] \;\subseteq\; y_0\,[\partial x \,|\, \partial t] \;+\; x_0\,[\partial y \,|\, \partial t]$*; where* $x_0 = x[t_0]$ *and* $y_0 = y[t_0]$.

Proof. This is essentially just a restatement of Theorems 6.4 and 6.7 in interval form, applying Definition 6.5 and Corollary 6.9. □

6.7 Inferring Flow Velocity Comparisons

Given two fluents, their flow velocity intervals provide some, but not complete, information on whether one fluent flows faster than the other through a given time-like instant. The next theorem makes this relationship more precise.

Theorem 6.11. *Suppose that* x *and* y *are two fluents on the same indexing interval* I*, with bounded flow velocities through* $t_0 \in I^o$*. Then the following relations hold through* t_0*:*

- *If* $x \lessdot y$*; then* $\grave{x} \leq \grave{y}$ *and* $\acute{x} \leq \acute{y}$.
- *Hence, if* $\grave{x} < \grave{y}$*, then* $\neg(x \gtrdot y)$*;*
- *and if* $\acute{x} < \acute{y}$*, then* $\neg(x \gtrdot y)$.
- *If* $\acute{x} < \grave{y}$*, then* $x \lessdot y$.
- *Hence, if* x *and* y *have unique flow velocity limits* $\dot{x}$ *and* $\dot{y}$ *through* t_0*, and* $\dot{x} < \dot{y}$*; then* $x \lessdot y$.

Proof. We prove the first and penultimate assertions (the others follow directly).

Suppose $x \lessdot y$ and $u < \grave{x}$. Then $u\psi_1 \lessdot x$ and hence also $u\psi_1 \lessdot y$, which implies $u \leq \grave{y}$; and this is true for all such $u < \grave{x}$. Therefore, $\grave{x} \leq \grave{y}$.

The dual result $\acute{x} \leq \acute{y}$ follows by considering $v > \acute{y}$ (or from the relation $(-y) \lessdot (-x)$).

If $\acute{x} < \grave{y}$, then $x \lessdot \frac{1}{2}(\acute{x} + \grave{y})\psi_1 \lessdot y$; so $x \lessdot y$. □

If we consider the flow velocity intervals $[\grave{x}, \acute{x}]$ and $[\grave{y}, \acute{y}]$ of two fluents $x[t]$ and $y[t]$ with bounded flow velocities, then the condition $\acute{x} < \grave{y}$ is equivalent to saying $[\partial x \,|\, \partial t] < [\partial y \,|\, \partial t]$. This in turn implies that $[\partial x \,|\, \partial t]$ and $[\partial y \,|\, \partial t]$ are disjoint.

Conversely, if x and y have bounded flow velocities, and $[\partial x \,|\, \partial t]$ and $[\partial y \,|\, \partial t]$ are disjoint; then either $x \lessdot y$ or $y \lessdot x$.

The next example shows that we can't infer whether one of the fluents flows faster than the other from their flow velocity intervals $[\partial x \,|\, \partial t]$ and $[\partial y \,|\, \partial t]$ alone when these two intervals overlap.

Example 6.8. We define three fluents x, y, z on the indexing interval $I = [-1, +1]$, where $[\grave{y}, \acute{y}] = [\grave{z}, \acute{z}]$; but $x \lessdot z$ and $\neg(x \lessdot y)$.

Define x as follows:

$$x[t] = \begin{cases} -2t & \text{if } t < 0 \\ t & \text{if } t \geq 0 \end{cases}$$

Define $y = -x$:

$$y[t] = \begin{cases} 2t & \text{if } t < 0 \\ -t & \text{if } t \geq 0 \end{cases}$$

Define $z = y^{\mathrm{op}}$:

$$z[t] = \begin{cases} -t & \text{if } t \leq 0 \\ 2t & \text{if } t > 0 \end{cases}$$

Then $[\grave{x}, \acute{x}] = [-2, +1]$ and $[\grave{y}, \acute{y}] = [\grave{z}, \acute{z}] = [-1, +2]$ through $t_0 = 0$.

6.8 Bounded Flow Velocity and Continuity

Within-bounds continuity of a fluent x at a time-like instant t_0 provides a constraint on how x can vary as it passes through t_0: it outlaws sudden jumps. Bounded flow velocity produces a different kind of constraint: it puts a finite limit on how fast any variation occurs, which would seem to be at least as strong, if not stronger. We might therefore ask whether one implies the other. The next theorem provides an affirmative reply.

We start with an alternative characterisation of bounded flow velocity, which is sometimes useful in mental visualisation and proofs.

Lemma 6.12. *A fluent x with indexing interval I has bounded flow velocity through $t_0 \in I^o$ iff there is a finite closed sub-interval $H \subseteq I$ straddling t_0 and a positive real constant $c > 0$ with:*

$$\big|x[t] - x[t_0]\big| \leq c\,|t - t_0| \text{ for all } t \in H.$$

Proof. Assume x has bounded flow velocity through t_0. Then $u\psi_1 \lessdot x \lessdot v\psi_1$ on H through t_0 for some closed sub-interval $H \subseteq I$ straddling t_0 and some $u, v \in \mathbb{R}$.

We can define c as generously as we wish. It suffices to put:

$$c = \max\{|u|, |v|\}.$$

The rest follows by separately considering intervals of form $J = [t, t_0]$ (where $t < t_0$) and $J = [t_0, t]$ (where $t > t_0$) for $t \in H$. In either case, $J \subseteq H$ and $|\,\Delta x|_J\,| < c\,\Delta J$, which expands to the required conclusion.

The converse follows after noting that:

$$-(1+c)\,\psi_1 \lessdot x \lessdot +(1+c)\,\psi_1 \quad \text{on } H \text{ through } t_0.$$

This completes the proof. ☐

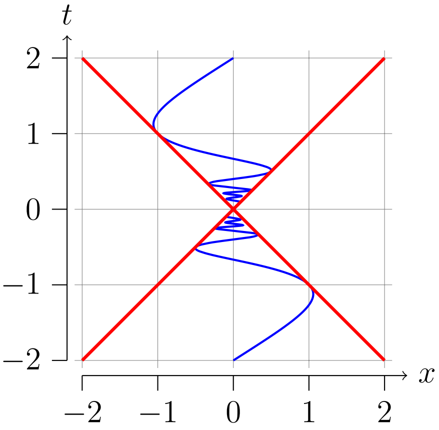

Figure 6.4: The fluent continuation $x_{\text{SNAKE}}[t]$ lies within the cone $|x - x_{\text{SNAKE}}[0]| \leq |t|$ for any finite closed interval straddling $t = 0$, so has bounded flow velocity through $t = 0$. (Sketched only.)

Example 6.9. The fluent $x_{\text{SNAKE}}[t]$ in Example 4.3 satisfies $|x_{\text{SNAKE}}[t]| \leq |t|$. It therefore has bounded flow velocity through $t = 0$, noting that by definition $x_{\text{SNAKE}}[0] = 0$.

This characterisation of bounded flow velocity might remind physicists of a light-cone in relativity (or at least one that's acting locally). If so, the following theorem is akin to telling us that if a moving object is constrained by the speed of light, then it can't undergo any sudden jumps in position. (Mathematicians may be reminded of Lipschitz continuity, which has a very similar cone criterion.)

Theorem 6.13. *If fluent x has bounded flow velocity through t_0, then it is also within-bounds continuous through t_0.*

Proof. Suppose x has indexing interval I straddling t_0; and construct $c > 0$ and $H \subseteq I$ straddling t_0 as in Lemma 6.12.

Now choose any $b > x[t_0]$. We wish to find a closed sub-interval $J \subseteq I$ straddling t_0 for which $x|_J < b$.

Put $\varepsilon = b - x[t_0] > 0$; and consider the interval $J = H \cap [t_0 - \delta, t_0 + \delta]$ where $\delta = \varepsilon/(1 + c)$. Then J straddles t_0; and for $t \in J$ with $t > t_0$:

$$\begin{aligned} x[t] - x[t_0] &\leq c(t - t_0) \\ &\leq c\delta \\ &< \varepsilon \end{aligned}$$

Hence $x[t] < \varepsilon + x[t_0] = b$. This inequality is true trivially for $t = t_0$, we've just proved it for any $t > t_0$, and a similar argument shows it's true for $t < t_0$; ie. it holds for all $t \in J$ as required.

The above construction works for all $b > x[t_0]$ and we can show similarly that it works for all $a < x[t_0]$. Therefore x is continuous through t_0, as claimed. $\square$

The proof derives from our definition of within-bounds continuity for a fluent$^{(\text{Def } 4.4)}$. Note that it is the value of the fluent that is constrained in this definition, rather than flow velocity, whose boundedness sets a constraint to changes in value. The resulting proof steps may seem slightly clumsy, which in part is because the definition of fluent continuity uses value-based bounds, mixing elements from different paradigms (arithmetic and kinematic).

An obvious question is whether bounded flow velocity is necessarily implied by within-bounds continuity. The next example shows that it isn't: it exhibits a fluent that is within-bounds continuous through a particular time-like instant, but doesn't have bounded flow velocity there.

Example 6.10. Consider the cube root fluent $x[t] = t^{1/3}$ on the indexing interval $I = [-1, +1]$, where I straddles the point $t_0 = 0$.

Given $b > x[0] = 0$, we see that $x[t]$ is strictly bounded above by b when we restrict t to the interval $I \cap [-1, +\frac{1}{2}b^3]$. In other words, for any such t, we then have $x[t] < b$. And similarly, it's strictly bounded below by any $a < 0$ on the interval $I \cap [\frac{1}{2}a^3, +1]$.

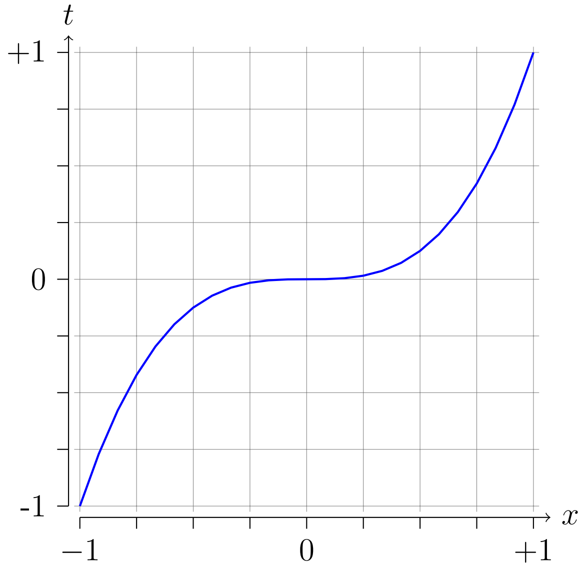

Figure 6.5: The fluent continuation $x[t] = t^{1/3}$ from Example 6.10, with indexing interval $I = [-1, +1]$.

It's therefore within-bounds continuous through $t_0 = 0$.

But there is no $v \in \mathbb{R}$ for which $x \lessdot v\psi_1$ through $t_0 = 0$, so x doesn't have bounded flow velocity there. To see this, note that $t = x^3$. Hence if $1 \geq h > 0$ and $J = [0, h]$ containing $t_0 = 0$, then $\Delta J = h = h^{2/3} h^{1/3} = h^{2/3} \Delta x|_J$; which can be rewritten as $\Delta x|_J = h^{-2/3} \Delta J$.

To establish $x \lessdot v\psi_1$ through $t_0 = 0$ requires a closed sub-interval $I^\ddagger = [T_1^\ddagger, T_2^\ddagger] \subseteq I$ straddling 0, where $\Delta x|_J < v \Delta J$ for each closed sub-interval $J \subseteq I^\ddagger$ containing 0. But we can always find $J = [0, h] \subseteq I^\ddagger$ with small enough $h > 0$ that $h^{-2/3} \geq v$, eg. by choosing $h = \min\{|v|^{-3/2}, T_2^\ddagger\}$.

Notice that a similar argument shows that $u\psi_1 \lessdot x$ fails through 0 for all $u \in \mathbb{R}$, so there is no finite lower velocity bound either (there's no sufficiently large $u \in \mathbb{R}$).

It would be tempting to write $\grave{x} = \acute{x} = \dot{x} = +\infty$ through 0 for this example fluent. We resist this temptation for now.

We can say that a fluent x of form $(\mathbb{R}, U, K, \nu)$ is *Lipschitz continuous* if its evaluation function $\nu \colon K \to U$ is Lipschitz continuous[9]. This means that, for some fixed $c > 0$ and every $t_1, t_2 \in K$, we have:

$$\left| x[t_2] - x[t_1] \right| \leq c \left| t_2 - t_1 \right| .$$

Equivalently, for some fixed $c > 0$ and every closed sub-interval $J \subseteq K$:

$$\left| \Delta x|_J \right| \leq c \, \Delta J.$$

If x is Lipschitz continuous on K and the closed sub-interval I is contained in K^o, then Lemma 6.12 implies that, for any $t_0 \in I$, x has bounded flow velocity through t_0. The next example shows the obvious converse fails.

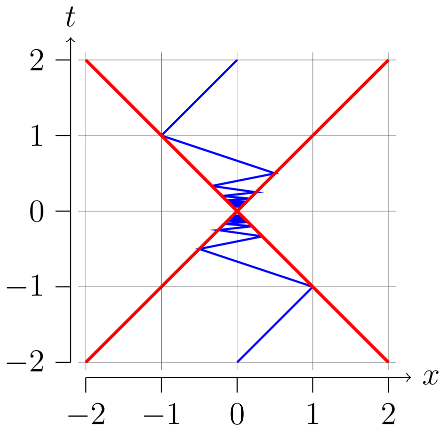

Figure 6.6: Similarly to $x_{\text{SNAKE}}[t]$, the fluent continuation $x_{\text{ZIGZAG}}[t]$ lies within the cone $|x - x_{\text{ZIGZAG}}[0]| \leq |t|$ for any finite closed interval straddling $t = 0$, so has bounded flow velocity through $t = 0$; but it is not Lipschitz continuous about zero. (Sketched only. Compare Figure 6.4)

Example 6.11. This example constructs a fluent x_{ZIGZAG} with indexing interval $I = [-1, +1]$, which has bounded flow velocity through each $t_0 \in I^o$, but is not Lipschitz continuous on any non-trivial finite closed interval containing $t_0 = 0$.

For each $k \in \mathbb{N} \setminus \{0\}$, set $s_k = \frac{1}{k}$. Define:

$$x_k = \begin{cases} +s_k & \text{if } k \text{ is even;} \\ -s_k & \text{if } k \text{ is odd;} \end{cases}$$

and define:

$$x_{\text{ZIGZAG}}[t] = \begin{cases} 0 & \text{if } t = 0; \\ \dfrac{s_k - t}{s_k - s_{k+1}} x_{k+1} + \dfrac{t - s_{k+1}}{s_k - s_{k+1}} x_k & \text{if } t \in \left]s_{k+1}, s_k\right]; \\ -x_{\text{ZIGZAG}}[-t] & \text{if } t < 0; \end{cases}$$

so $x_{\text{ZIGZAG}}[s_k] = x_k$, for each k.

As an aside, notice that this can be extended fairly naturally to the whole of $\mathbb{R}$ by defining $x_{\text{ZIGZAG}}[t] = t - 2$ when $t > 1$ (equivalently $t \in \left]1, +\infty\right[$). It is then a piecewise linear counterpart to the fluent extension $x_{\text{SNAKE}}[t]$ of Example 4.3.

To show x_{ZIGZAG} has the desired properties, consider the segment corresponding to the parametric interval $[s_{k+1}, s_k]$. Put $\Delta t = s_k - s_{k+1}$ and $\Delta x = x_{\text{ZIGZAG}}[s_k] - x_{\text{ZIGZAG}}[s_{k+1}]$.

Then:

$$\begin{aligned} & \Delta t = \frac{1}{k} - \frac{1}{k+1} = \frac{1}{k(k+1)}; \\ & \Delta x = \pm\left(\frac{1}{k} + \frac{1}{k+1}\right) = \pm\left(\frac{2k+1}{k(k+1)}\right); \\ \implies \quad & \Delta x = \pm(2k+1)\Delta t. \end{aligned}$$

One can use this result to show that x_{ZIGZAG} has bounded flow velocity through each $t_0 \in I^o$ by considering the three separate cases:

- $t_0 \in \,]s_{k+1}, s_k[$;
- $t_0 = s_k$, for $k > 1$; and
- $t_0 = 0$, for which $|x_{\text{ZIGZAG}}[t] - x_{\text{ZIGZAG}}[t_0]| \leq |t - t_0|$.

Moreover, given $c > 0$, we can always find sufficiently large k so that $|\Delta x| > c|\Delta t|$; so x_{ZIGZAG} isn't Lipschitz continuous on I or on any non-trivial finite closed interval containing 0.

6.9 Mean Value Theorem

The Mean Value Theorem[11] is a classic result in analysis. It has its own parallel for fluents with bounded flow velocity, whose development is sketched in this section.

Definition 6.6. Suppose x is a real-valued fluent with indexing interval I. Fluent x has a *local minimum through* $t_0 \in I^o$ iff there is a closed finite sub-interval $H \subseteq I$ straddling t_0 for which $x|_H \geq x[t_0]$.

Dually, fluent x has a *local maximum through* $t_0 \in I^o$ iff there is a closed finite sub-interval $H \subseteq I$ straddling t_0 for which $x|_H \leq x[t_0]$.

Remark 6.6. The minimum of $x|_I$ is trivially also a local minimum if it occurs through $t_0 \in I^o$; and a similar result holds for the maximum.

Corollary 6.14. *If fluent x has a local minimum or local maximum through t_0, then $\neg(x \lessdot 0)$ and $\neg(x \gtrdot 0)$ through t_0.*

Proof. By the third assertion of Theorem 5.5, if $x \gtrdot 0$ through t_0, then it can't have a local minimum or local maximum through t_0; and similarly for $x \lessdot 0$. The result follows. ☐

Theorem 6.15. *Suppose x is a real-valued fluent with indexing interval I and bounded flow velocity through $t_0 \in I^o$. If fluent x has a local minimum or local maximum through t_0, then $0 \in [\partial x \,|\, \partial t]$ through t_0.*

Proof. Assume fluent x has a local minimum or local maximum through t_0. Then, by the previous corollary, $\neg(x \lessdot 0)$ and $\neg(x \gtrdot 0)$ through t_0.

Applying Theorem 6.2, then $\grave{x} \leq 0 \leq \acute{x}$. Hence the result. ☐

Example 6.12. Consider the fluent $x = |\psi_1|$ defined on $I = [-2, +2]$. Take $t_0 = 0 \in I^o$, where x has a (local) minimum. Fluent x has flow velocity interval $[\partial x \,|\, \partial t] = [-1, +1]$ through $t_0 = 0$, and so $0 \in [\partial x \,|\, \partial t]$ as claimed.

Theorem 6.16 (Rolle's Theorem for Fluents)**.** *Suppose x is a real-valued fluent with indexing interval I and bounded flow velocity through t for all $t \in I^o$. Suppose also that H is a non-trivial closed finite sub-interval $H \subseteq I^o$ with $\Delta x|_H = 0$.*

Then, for some $t_0 \in H$, we have $0 \in [\partial x \,|\, \partial t]$ through t_0.

Proof. Fluent x has bounded flow velocity through each $t \in H$, so is within-bounds continuous there. Appealing to classical analysis and the equivalence of within-bounds continuity with classical continuity (or by first principles), we can show by the Extreme Value Theorem[6] that $x|_H$ attains its minimum and maximum at one or more points within H.

If both the minimum and maximum are attained at either $\min H$ or $\max H$, then this must be so for both $\min H$ and $\max H$, and $x|_H$ will be constant. In this case, take $t_0 = \frac{1}{2}(\min H + \max H)$, for which trivially $0 \in [\partial x \,|\, \partial t]$ through t_0.

Otherwise, $x|_H$ has a minimum or maximum through some $t_0 \in H^o$, for which $0 \in [\partial x \,|\, \partial t]$ through t_0 by the previous theorem.

This proves the claim for all possible cases. ☐

Theorem 6.17 (Mean Value Theorem I)**.** *Suppose x is a real-valued fluent with indexing interval I and bounded flow velocity through t for all $t \in I^o$. Suppose also that H is a non-trivial closed finite sub-interval $H \subseteq I^o$. Then, for some $t_0 \in H^o$:*

$$\frac{\Delta x|_H}{\Delta H} \in [\partial x \,|\, \partial t] \;\; \textit{through } t_0.$$

Proof. Define a new fluent z on I by:

$$z[t] = x[t] - \frac{\Delta x|_H}{\Delta H} \psi_1[t - t_1],$$

for all $t \in I$.

Then $z[t_2] = z[t_1]$, and z and H satisfy the conditions for Rolle's theorem. So, for some $t_0 \in H$, we have $0 \in [\partial z \,|\, \partial t] = [\partial x \,|\, \partial t] - \frac{\Delta x|_H}{\Delta H}$ through t_0.

Hence the result. ☐

Example 6.13. With the same notation and conditions as Theorem 6.17, suppose that, for some $c \geq 0$, x satisfies $[\partial x \,|\, \partial t] \subseteq [-c, +c]$ through each $t_0 \in I^o$. Then this theorem implies that for every closed sub-interval $J \subseteq H$:

$$\big|\Delta x|_J\big| \leq c\,\Delta J.$$

In other words, $x|_H$ is Lipschitz continuous.

Theorem 6.18 (Mean Value Theorem II). *Suppose x is a real-valued fluent with indexing interval I and has unique flow flow velocity limits through t for all $t \in I^o$. Suppose also that H is a non-trivial closed finite sub-interval $H \subseteq I^o$. Then, for some $t_0 \in H^o$:*

$$\frac{\Delta x|_H}{\Delta H} = \dot{x}|_{t_0}.$$

Proof. This follows immediately from the preceding theorem. □

Remark 6.7. The previous definition and accompanying results apply in an obvious way if x is a fluent continuation.

6.10 Comparison with Classical Differentiation

This section provides an informal comparison between fluent flow velocity and the classical time derivative.

6.10.1 Similarities and Differences

There are obvious similarities between a fluent $x[t]$ and the corresponding function of physical time $x(t)$. In the cases when the fluent x has a unique flow velocity limit through some time-like instant t_0, which therefore satisfies $\grave{x} = \acute{x} = \dot{x}$ through t_0, this leads to a natural urge to equate $\dot{x}$ with the classical time derivative $\frac{dx}{dt}$. This latter similarity is borne out by our previous results, such as the formulae in Table 6.1 and the product rule. Another parallel result is stated in Theorem 6.13, where we proved the continuity of fluents with bounded flow velocity and hence those with a unique flow velocity limit.

There are, however, contrasts. The most obvious is that classical differentiation is usually forced to treat the abrupt turn of the modulus function $f(t) = |t|$ at $t = 0$ as a special case; whereas it fits quite naturally within our analysis of fluents with bounded flow velocity. Another is conceptual.

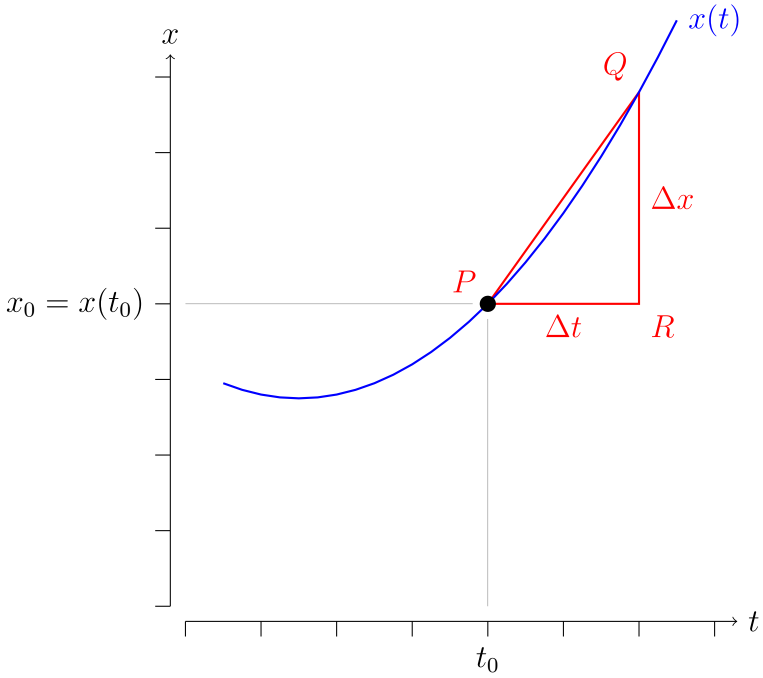

Figure 6.7: As the sides of the differential triangle PQR decrease towards zero, keeping P fixed, the secant PQ approaches the tangent at $P = (t_0, x_0)$, where $x_0 = x(t_0)$; and the quotient $\Delta x / \Delta t$ approaches the gradient $\frac{dx}{dt}$.

6.10.2 Conceptual Differences

The classical approach is often illustrated using a differential triangle, which was known to Newton's predecessors, notably Isaac Barrow. See Figure 6.7.

The claim is then that, as the base of the triangle Δt becomes arbitrarily small, the quotient $\Delta x / \Delta t$ becomes arbitrarily close to the gradient of the tangent at $t = t_0$, and hence to the instantaneous velocity there when x represents a displacement. We usually implicitly assume that x is well-behaved near the point of interest $P = (t_0, x_0)$, for $x_0 = x(t_0)$; eg. that it's differentiable (often continuously so) over a neighbourhood of P.

The fluent-based equivalent (as developed here, rather than by Newton) would be somewhat different, as depicted in Figure 6.8.

For ease of comparison, in this case the fluent diagram shows time along the horizontal axis. It illustrates the condition $u\psi_1 \lessdot x \lessdot v\psi_1$ through $t = t_0$. The fluent lines $\lambda_u = u\,(\psi_1 - t_0) + x_0$ and $\lambda_v = v\,(\psi_1 - t_0) + x_0$ bracket the gradient of the tangent through P between the lower and upper flow velocity test-values u and v. Note that $u\psi_1 \lessdot x \lessdot v\psi_1$ if and only if $\lambda_u \lessdot x \lessdot \lambda_v$.

If we assume that x has bounded flow velocity through t_0 in this way, then

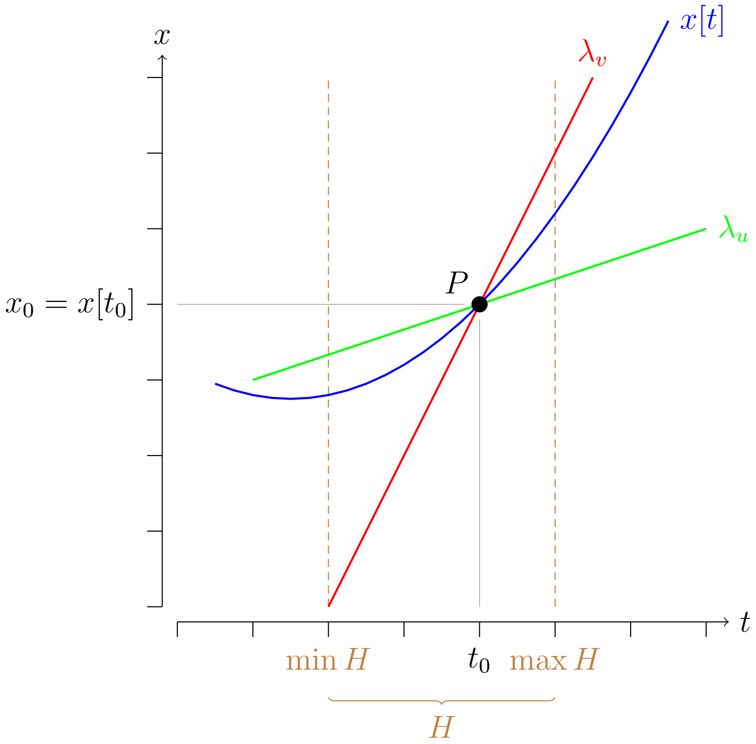

Figure 6.8: A fluent $x[t]$ with bounded flow velocity is bracketed within interval H by a skew cone made up of linear fluents λ_u and λ_v through $P = (t_0, x_0)$, which correspond to uniform flow velocities u and v. As the gap between u and v gets tighter, the angle between lines λ_u and λ_v decreases towards zero. (Compare Figures 8.2 and 6.7.)

we're able to prove this bracketing property (which isn't generally available when appealing to differential triangles). Indeed, the algebraic relations in Corollary 6.1 describe what we see in the diagram. Moreover, the angle between λ_u and λ_v will decrease if we choose u closer to $\grave{x}$ and v closer to $\acute{x}$ (which may then require a new H).

Visualisation using bracketing lines of uniform flow velocity works for even quite pathological fluents, such as the fluent $(\psi_1^2\, 1_{\mathbb{Q}})[t]$, which takes the value t^2 when t is rational and is zero otherwise. Treated as a function, this is differentiable classically at $t = 0$, but isn't usually what one expects to deal with when arguing visually with differential triangles. Example 7.1 discusses this fluent.

Bracketing of fluent flow velocity also works for the fluent $|\psi_1|[t] = |t|$, where differential triangles might struggle, despite the modulus function being both common and relatively benign.

6.10.3 Non-equivalence of Velocity Comparison

It would be tempting to try to express the ‘flows faster than’ relation in terms of derivatives. The obvious formulation for x and y both differentiable at t_0 would be that “x flows faster than y through t_0 iff the derivative of x exceeds the derivative of y at $t = t_0$”. In practice, it doesn’t work like this for fluents.

For example, when $x[t] = x(t) = t^3$ the equivalent of the derivative at $t = 0$ is $\grave{x} = \acute{x} = \dot{x} = 0$ through $t_0 = 0$; but the fluent x flows faster than $0\psi_1$ through t_0, despite the corresponding derivatives both being 0 there. The fluent relation ‘$\Box$ flows faster than $\Box$ through $\Box$’ is therefore something different.

6.10.4 Existence of an Instantaneous Velocity

Although $\dot{x}$ associates a specific time-like instance with a well-defined (and unique) velocity value, it isn’t entirely clear (to me, at least) that we’re justified in considering $\dot{x}$ to be the instantaneous velocity of a physical object represented by the fluent $x[t]$, assuming of course that such an object exists.

The development here differs from modern analysis, which asks us to accept the existence of an instantaneous velocity for a displacement function such as $x(t) = t^2$ and claims that it can be arrived at by a limiting process of arbitrarily small time increments, or perhaps a rigorous theory of infinitesimals.

Our fluent theory can afford to be non-committal about instantaneous velocity, because the theory is founded on the comparison of flow velocity, which only ever requires that the time intervals involved be of non-zero length. It’s this agnosticism that enables it to be unfazed when fluents such as $|\psi_1|[t]$ are involved in these comparisons, for which an instantaneous velocity through the instant $t = 0$ would make no sense (or at least it makes little sense to claim there must be a unique one).

6.10.5 Differentiating by Analogy

Having at this stage produced something that looks very like a classical time derivative, we might (with reason) be tempted to justify differentiation of a general function (such as $f(x) = x^3$) by direct analogy with differentiation with respect to time (eg. of $x(t) = t^3$). There is, however, a final conceptual distinction in founding differential calculus on fluent flow velocity, rather than the classical time derivative. If not, we could appeal to fluent flow velocity as the basis of our analogy or simply change our terminology to suggest more general rates of change. Instead, I’ll pursue a somewhat different approach, which is closer to Newton’s definition of a function derivative as a ratio of two fluxions (one side effect being that it renders the chain rule more or less for free).

Conceptually, fluents permit the possibility of replacing one parametrisation by another, which makes little sense in a classical function derivative. We exploit this freedom when we construct the derivative of a function in Chapter 8 .

Following Newton's formulation, when $y = f(x)$, $x_0 = x[t_0]$ and the fluent $x[t]$ satisfies $\dot{x}|_{t_0} \neq 0$, Chapter 8 tells us that:

$$\left.\frac{dy}{dx}\right|_{x_0} \equiv \frac{\dot{y}|_{t_0}}{\dot{x}|_{t_0}},$$

where the right hand expression is an ordinary fraction.

Furthermore, we can always define $x[t] = t$ and put $x_0 = x[t_0] = t_0$, so that $\dot{x}|_{t_0} = 1$. This equation then yields:

$$\left.\frac{dy}{dx}\right|_{x_0} = \dot{y}|_{t_0} .$$

Instead of arguing by analogy, the resemblance between the classical function derivative and an equivalent fluent flow velocity construction emerges as a special case of the more general quotient between two flow velocity limits.

Part III

Fluxions and Derivatives

Chapter 7

Fluxions

Some modern texts equate the fluxion with a time derivative, but Newton originally defined a derivative as the ratio between two fluxions, which is the approach embraced here. Many of the properties of fluxions follow directly from those of fluents with bounded flow velocity, so this chapter is a fairly short one.

I'm not suggesting the approach in this chapter (or indeed this document) is one which Newton took. On the contrary, it's only loosely inspired historically, while aspiring to modern standards of rigour mathematically.

7.1 Definition as a Derived Fluent

We defined the greatest lower and least upper flow velocities of a fluent x through a time-like instant t_0 in Chapter 6. When these two values were equal, we denoted their common value by $\dot{x}$ or $\dot{x}|_{t_0}$. We now try to take this further. If we can apply this same construction to every t_0 in an indexing interval I of x, then the values of $\dot{x}|_{t_0}$ describe a new fluent on I, which we can also conveniently denote by $\dot{x}$, with very little ambiguity.

This leads us to the following definition, where I've introduced the containing interval K to ensure that we can find suitable test intervals straddling the time-like instants $t_0 = \min I$ and $t_0 = \max I$.

Definition 7.1. Suppose x is a fluent on an indexing interval K; and that $I \subseteq K^o$ is a non-trivial closed sub-interval, contained within the interior of K. Suppose also that, for each $t_0 \in I$, x has bounded flow velocity through t_0 and $\grave{x} = \acute{x} = \dot{x}$ through t_0.

Then for each $t \in I$ we set $\dot{x}[t] = \dot{x}|_t$, defining a new fluent $\dot{x}$ on indexing interval I. This new fluent is the *fluxion on I derived from x*.

Remark 7.1. If the same process can be repeated to get the fluxion of a fluxion, we will create a second-order fluent denoted $\ddot{x}[t]$.

Remark 7.2. It is relatively straightforward to replace K in this definition by an open interval D, so that x becomes a fluent continuation on D. The derived fluent can then be defined on any non-trivial closed sub-interval $I \subseteq D$ straddling any given $t_0 \in D$. The effect is then very similar to the classical approach to forming a time derivative throughout an open interval.

We might typically construct a fluxion from a given fluent using the results from Chapter 6, perhaps resorting to first principles. In the general case, there's no guarantee that the fluxion exists on the whole of I. For example, it may fail because $\dot{x}|_{t_0}$ is well-defined for a particular t_0, but not for its neighbours in I. This prompts a less stringent definition.

Suppose x is a fluent with indexing interval K straddling $t_0 \in K^o$, and x has the unique flow velocity limit $\dot{x}$ through t_0. Then we may informally call the value $\dot{x}|_{t_0}$ the *fluxion at* t_0 *derived from* x (or just the *fluxion at* t_0), even if we haven't established that the fluxion exists as a fluent on the entirety of a non-trivial closed sub-interval $I \subseteq K$.

Very often, we needn't specify whether we mean 'fluxion' as a well-defined fluent or solely as a spot value, because either the context will be obvious or the distinction won't matter. The formal name *unique flow velocity limit* can be used in place of fluxion when a spot value is intended, which I've tended to use in this document for clarity.

It can be convenient to have an equivalent operator form for the fluxion, eg. $\operatorname{flx} x \equiv \dot{x}$. It can be used by extension for the unique flow velocity limit, as in $(\operatorname{flx} x)|_{t_0} \equiv \dot{x}|_{t_0}$.

Example 7.1. The fluent continuation $1_{\mathbb{Q}}[t]$ equals 1 when t is rational and 0 otherwise. It's highly discontinuous and fails to have bounded flow velocity anywhere, let alone qualify as a fluxion. Its graph can be thought of as a toothcomb; and it can be explicitly expressed as follows:

$$1_{\mathbb{Q}}[t] = \begin{cases} 1 & t \in \mathbb{Q}; \\ 0 & \text{otherwise.} \end{cases}$$

We can moderate it slightly to form the the fluent continuation $q = {\psi_1}^2\, 1_{\mathbb{Q}}$, which satisfies:

$$q[t] = \begin{cases} t^2 & t \in \mathbb{Q}; \\ 0 & \text{otherwise.} \end{cases}$$

Notice that $1_{\mathbb{Q}}[t]^2 = 1_{\mathbb{Q}}[t]$, so that $q[t] = \tau^2$, where $\tau = t\, 1_{\mathbb{Q}}[t]$. Then for any interval $J = [t_1, t_2]$, we have $\Delta q|_J = (\tau_1 + \tau_2)\,(\tau_2 - \tau_1)$, where $\tau_i = t_i\, 1_{\mathbb{Q}}[t_i]$ for each $i \in \{1, 2\}$.

Now suppose that J contains $t_0 = 0$, so that $t_1 \le 0 \le t_2$. Then we must have $\tau_2 - \tau_1 \le t_2 - t_1 = \Delta J$; so $\Delta q|_J \le (\tau_1 + \tau_2)\Delta J \le 2 \max\{\tau_1, \tau_2\}\Delta J$. Hence, given

$v > 0$, we can choose a tight enough closed interval $I^\ddagger$ straddling $t_0 = 0$ so that $q \lessdot v\psi_1$ on $I^\ddagger$ through t_0.

The dual result holds for any test velocity $u < 0$. It follows that $\grave{q} = \acute{q} = 0$ through $t_0 = 0$, so the fluxion $\dot{q}|_0$ at $t_0 = 0$ exists and is zero.

But when $t_0 > 0$, for any $v > 0$ we can choose a purposely narrow $J = [t_1, t_2]$ containing t_0, where $\Delta J < t_0^2/v$ and $t_2 \in \mathbb{Q}$ and $t_1 \notin \mathbb{Q}$. In that case, $\Delta q|_J = t_2^2 \geq t_0^2 > v\Delta J$; so q doesn't have bound flow velocity, let alone a well-defined value for $\dot{q}|_{t_0}$ for $t_0 > 0$. A similar result is true for $t_0 < 0$, and hence for all $t_0 \neq 0$.

7.2 Basic Properties

If a fluent x has a corresponding fluxion, then x must have bounded flow velocity and the results of Chapter 6 will apply. The following properties are simple specialisations, exploiting the equality of $\dot{x} = \grave{x} = \acute{x}$.

Theorem 7.1. *Suppose $a \in \mathbb{R}$; and that x and y are two real-valued fluents defined on a common indexing interval K. Assume also that the fluxions $\dot{x}$ and $\dot{y}$ are well-defined on $I \subseteq K^o$ (ie. $\dot{x}|_{t_0}$ and $\dot{y}|_{t_0}$ exist through each $t_0 \in I \subseteq K^o$). Then the following basic relations also hold.*

1. *(Duality) If fluent $w = \operatorname{opp} x$, then the fluxion $\dot{w}$ exists on I and $\dot{w} = -\dot{x}$.*
2. *(Translational invariance) If fluent $z = x + a$, then $\dot{z}$ exists on I and $\dot{z} = \dot{x}$.*
3. *(Scaling) If fluent $z = ax$, then $\dot{z}$ exists on I and $\dot{z} = a\dot{x}$.*
4. *(Summation) If fluent $z = x + y$, then $\dot{z}$ exists on I and $\dot{z} = \dot{x} + \dot{y}$.*
5. *(Product rule for fluxions) If fluent $z = xy$, then $\dot{z}$ exists on I and $\dot{z} = \dot{x}y + x\dot{y}$.*

Proof. Each assertion follows from Theorem 6.4 for each $t_0 \in I$; except the Product rule, which derives from Theorem 6.7. Fix $t_0 \in I$ and take Summation by way of example.

If $z = x + y$, then Summation 2 from Theorem 6.4 implies that z has bounded flow velocity and $\grave{x} + \grave{y} = \grave{z} \leq \acute{z} = \acute{x} + \acute{y}$.

But $\grave{x} = \acute{x} = \dot{x}$; therefore $\dot{x} + \dot{y} = \grave{z} \leq \acute{z} = \dot{x} + \dot{y}$. Hence $\grave{z} = \acute{z} = \dot{x} + \dot{y}$ and the unique flow velocity limit $\dot{z}$ exists through t_0, with $\dot{z} = \dot{x} + \dot{y}$ as claimed.

A similar sandwiching argument applies to Duality, Translational invariance, Scaling and the Product rule. $\square$

Remark 7.3. We've used the dot-notation $\dot{x}$, $\dot{y}$, etc, here to denote the fluxion on the whole of I.

Nevertheless, notice that if $\dot{x}$ and $\dot{y}$ were well-defined through a given $t_0 \in I$, then each of the above assertions would hold through t_0. This is so, even if these pre-conditions didn't apply to the whole of I.

One might wish to extend the analogy between a fluent's time-like parameter and physical time to one between the fluxion and physical velocity. If so, the following theorem is in keeping with this comparison.

Theorem 7.2. *Suppose that x is a real-valued fluent defined on the indexing interval K. Assume also that the fluxion $\dot{x}$ exists on the non-trivial closed sub-interval $I \subseteq K$. Then the following propositions are true.*

1. *If $\dot{x} > 0$, then $x \gtrdot 0$ on I; which in turn implies that x is strictly increasing on I.*

2. *If $\dot{x} < 0$, then $x \lessdot 0$ on I; so x is strictly decreasing on I.*

3. *If $\dot{x} = 0$, then x is constant on I.*

Proof. Take each proposition in order.

1. If $\dot{x} > 0$, then $0 < \grave{x}[t] = \dot{x}[t]$ for every $t \in I$. So $0 \lessdot x$ through t for each $t \in I$; and by Lemma 5.7, $0 \lessdot x$ on I. This then implies x is strictly increasing on I by Theorem 5.2.

2. The case of $\dot{x} < 0$ is the obvious dual result.

3. Suppose $\dot{x} = 0$. We assume that x isn't constant on I and derive a contradiction.

   First, suppose we can find $t_1 < t_2$ within I, where $x[t_1] < x[t_2]$. Then the non-trivial interval $J = [t_1, t_2]$ satisfies $\Delta x|_J = v\,\Delta J$, where $v = \Delta x|_J \,/\, \Delta J > 0$.

   But $\dot{x}[t] = 0$ implies for all $v > 0$ that $x \lessdot v\psi_1$ through t; and this is true of each $t \in I$. Hence $x \lessdot v\psi_1$ on I. Therefore J is a permissible test interval, implying that $\Delta x|_J < v\,\Delta J$, which is a contradiction. ※

   Similarly, it can't be that $x[t_1] > x[t_2]$.

   Hence we must have $x[t_1] = x[t_2]$; and this must be true of all pairs t_1 and t_2 in I, ie. $x|_I$ must be constant.

This concludes the proof. □

Remark 7.4. The fluent continuation $x[t] = t^3$ shows that it's possible for $x \gtrdot 0$ through all $t_0 \in I$ (eg. for $I = [-1, +1]$), but it isn't the case that $\dot{x}[t_0 = 0] > 0$, so we need to be careful with the converses of these assertions.

We can, however, say that $x \gtrdot 0$ through all $t \in I$ implies that $\dot{x} \geq 0$ there. We can also be sure by the previous theorem that we can't have $\dot{x} = 0$ on a non-trivial sub-interval $H \subseteq I$.

Chapter 8

Derivative of a Function

8.1 Notation

It's helpful in this chapter to notationally distinguish between fluents and plain variables. I'll typically denote a plain variable as used in a function by an undecorated letter, eg. plain x and y. Typically, y will be a function of x, written $y = f(x)$.

Where it seems helpful, I'll denote a fluent by a superscript, such as x^{A}, x^{B} or $x^{\star}$; or similarly y^{A}, etc. Superscripts can also be useful to distinguish other fluent-related objects, such as distinct indexing intervals I^{A} and I^{B}.

8.2 Preamble

We saw in Chapter 3 that given a real function $f\colon U \to \mathbb{R}$ and a real-valued fluent x^{A} defined on indexing interval I^{A}, then we could construct a fluent $y^{\mathsf{A}} = f \circ x^{\mathsf{A}}$ on I^{A} by function composition. In this case, $y^{\mathsf{A}}[t^{\mathsf{A}}] = f(x^{\mathsf{A}}[t^{\mathsf{A}}])$ for each $t^{\mathsf{A}} \in I^{\mathsf{A}}$ and we might ask what other fluent properties are carried across by f.

We start by exploring this question informally to nurture our intuition. In particular, we wish to characterise y^{A}'s flow velocity for a particular f, given that x^{A} has bounded flow velocity or, stronger, that the fluxion $\dot{x}^{\mathsf{A}}$ exists on I^{A}.

Suppose that x^{A} has bounded flow velocity through $t_0^{\mathsf{A}} \in I^{\mathsf{A}}$ and we establish that $y^{\mathsf{A}} = f \circ x^{\mathsf{A}}$ has bounded flow velocity through t_0^{A} as well. Then $x^{\mathsf{A}} \lessdot u\psi_1$ for some u and $y^{\mathsf{A}} \lessdot v\psi_1$ for some v, which means that for any qualifying test interval $J^{\mathsf{A}} \subseteq I^{\mathsf{A}}$ we have:

$$\Delta\, x^{\mathsf{A}}|_{J^{\mathsf{A}}} < u\,\Delta J^{\mathsf{A}}$$

and

$$\Delta\, y^{\mathsf{A}}|_{J^{\mathsf{A}}} < v\,\Delta J^{\mathsf{A}}.$$

The values u and v in these inequalities depend on the choice of parametrisation, so suppose now that we choose a second time-like parameter t^{B} over an indexing interval I^{B}, where the t^{A}-interval $J^{\mathsf{A}} \subseteq I^{\mathsf{A}}$ corresponds to a t^{B}-interval $J^{\mathsf{B}} \subseteq I^{\mathsf{B}}$. More specifically, we require that x^{A} and its counterpart x^{B} have the same start and end-points in U across the respective time-like intervals J^{A} and J^{B}; and, since $y = f(x)$, the fluents y^{A} and y^{B} will likewise share the same start and end-points in $\mathbb{R}$ across J^{A} and J^{B}.

In that case,

$$\Delta\, x^{\mathsf{B}}|_{J^{\mathsf{B}}} = \Delta\, x^{\mathsf{A}}|_{J^{\mathsf{A}}}$$

and

$$\Delta\, y^{\mathsf{B}}|_{J^{\mathsf{B}}} = \Delta\, y^{\mathsf{A}}|_{J^{\mathsf{A}}},$$

where x^{B} and $y^{\mathsf{B}} = f \circ x^{\mathsf{B}}$ denote the reparametrised fluents.

There's no guarantee that $\Delta J^{\mathsf{A}} = \Delta J^{\mathsf{B}}$, so we next imagine a change of scale in our choice of t^{B} such that numerically:

$$\Delta J^{\mathsf{A}} = 2\, \Delta J^{\mathsf{B}}.$$

A notional observer would then consider that x^{B} travels the same net displacement in value $\Delta x^{\mathsf{A}}|_{J^{\mathsf{A}}} = \Delta x^{\mathsf{B}}|_{J^{\mathsf{B}}}$ in twice the time measured in t^{A}-units than in t^{B}-units.

Hence, x^{B} will appear (on average) to flow at twice the velocity relative to t^{B} than x^{A} under t^{A}; and, because the equation $\Delta J^{\mathsf{A}} = 2\, \Delta J^{\mathsf{B}}$ determines y^{A} and y^{B}'s relative flow velocity as well, y^{B} will also appear to flow at twice the velocity under t^{B} than y^{A} under t^{A}.

One might therefore expect:

$$x^{\mathsf{B}} \lessdot 2u\psi_1 \quad \text{through } t_0^{\mathsf{B}}$$

and

$$y^{\mathsf{B}} \lessdot 2v\psi_1 \quad \text{through } t_0^{\mathsf{B}},$$

where we choose $t_0^{\mathsf{B}} \in I^{\mathsf{B}}$ to satisfy:

$$x^{\mathsf{B}}[t_0^{\mathsf{B}}] = x^{\mathsf{A}}[t_0^{\mathsf{A}}],$$

and hence:

$$y^{\mathsf{B}}[t_0^{\mathsf{B}}] = y^{\mathsf{A}}[t_0^{\mathsf{A}}].$$

Of course, there's nothing special about the factor of 2 in this example: we could replace 2 by any suitable factor $k \in \mathbb{R}$ and apply the same informal reasoning to the equation $\Delta J^{\mathsf{A}} = k\, \Delta J^{\mathsf{B}}$. This would then lead us to expect $x^{\mathsf{B}} \lessdot ku\psi_1$ through t_0^{B} and $y^{\mathsf{B}} \lessdot kv\psi_1$ through t_0^{B}, provided the various quantities involved were positive or at least non-negative.

Now consider what happens when all the necessary fluxions in x and y exist. We can apply the same reasoning to each $u > \dot{x}^{\mathsf{A}}[t^{\mathsf{A}}]$ and $v > \dot{y}^{\mathsf{A}}[t^{\mathsf{A}}]$ at each $t^{\mathsf{A}} \in I^{\mathsf{A}}$; and dually to $u < \dot{x}^{\mathsf{A}}[t^{\mathsf{A}}]$ and $v < \dot{y}^{\mathsf{A}}[t^{\mathsf{A}}]$.

Following this through suggests the hypothesis that, for a suitably behaved function f, we will have $\dot{x}^{\mathsf{B}} = k\,\dot{x}^{\mathsf{A}}$ and $\dot{y}^{\mathsf{B}} = k\,\dot{y}^{\mathsf{A}}$ at each $t_0^{\mathsf{A}} \in I^{\mathsf{A}}$ and corresponding $t_0^{\mathsf{B}} \in I^{\mathsf{B}}$. Ignoring inconvenient details, this then implies our key insight. We expect that the ratio:

$$\dot{x}^{\mathsf{A}} : \dot{y}^{\mathsf{A}} \quad \text{at } t_0^{\mathsf{A}}$$

will equal that of:

$$\dot{x}^{\mathsf{B}} : \dot{y}^{\mathsf{B}} \quad \text{at } t_0^{\mathsf{B}}.$$

In other words, we can expect to derive the *fundamental fluxion identity*:

$$\dot{x}^{\mathsf{A}}\dot{y}^{\mathsf{B}} = \dot{x}^{\mathsf{B}}\dot{y}^{\mathsf{A}} \quad \text{through } t_0^{\mathsf{A}} \text{ and } t_0^{\mathsf{B}}.$$

Isaac Newton described a similar insight, in his case arguing from a standpoint of “indefinitely small” time increments.[3] We build on our own model of fluents and fluxions in what follows.

This preamble has been purposely simplistic to build intuition and direction. A more rigorous argument needs to at least address the following.

- We’ve considered the equality $\Delta J^{\mathsf{A}} = k\,\Delta J^{\mathsf{B}}$ as if it holds for all eligible pairs J^{A} and J^{B}. This is too strong a relational constraint to be relied on in practice.
- The equality $\Delta\, x^{\mathsf{B}}|_{J^{\mathsf{B}}} = \Delta\, x^{\mathsf{A}}|_{J^{\mathsf{A}}}$ glosses over a potential change of sign.
- To derive the full fundamental fluxion identity, we need to expand the ratio argument to accommodate cases where one (or more than one) of the flow velocity limits involved is zero; eg. if $\dot{x}^{\mathsf{B}}[t_0^{\mathsf{B}}] = 0$.

We address all these points in what follows, eventually producing a more rigorous statement of the fundamental fluxion identity.

Remark 8.1. As a notational aside, one might wonder whether we should be applying the A and B superscripts to the fluent continuation ψ_1 when comparing our flow velocities against unit flows. For example, should we write $x^{\mathsf{B}} \lessdot 2u\psi_1^{\mathsf{B}}$? This would emphasise that ψ_1^{B} is parametrised by t^{B} in this relation, where $\psi_1^{\mathsf{B}}[t^{\mathsf{B}}] = t^{\mathsf{B}}$. In the current context, the decorated forms for ψ_1 are unnecessary. The fluent continuation is $\psi_1 = (\mathbb{R}, \mathbb{R}, \mathbb{R}, \mathrm{id}_{\mathbb{R}})$, where $\forall t \in \mathbb{R}\,(\mathrm{id}_{\mathbb{R}}(t) = t)$. The decorated versions would just be restrictions of the same continuation with $\psi_1^{\mathsf{A}} = \psi_1|_{I^{\mathsf{A}}}$ and $\psi_1^{\mathsf{B}} = \psi_1|_{I^{\mathsf{B}}}$; and the resulting fluents would be equal on the intersection $I^{\mathsf{A}} \cap I^{\mathsf{B}}$ of their indexing intervals.

In contrast, x^{A} and x^{B} are two independent fluents (other than both having values in $U \subseteq \mathbb{R}$) and take the form $x^{\mathsf{A}} = (\mathbb{R}, U, I^{\mathsf{A}}, \nu^{\mathsf{A}})$ and $x^{\mathsf{B}} = (\mathbb{R}, U, I^{\mathsf{B}}, \nu^{\mathsf{B}})$. If $I^{\mathsf{A}} \cap I^{\mathsf{B}} \neq \varnothing$, there is no guarantee the two fluents will be equal on $I^{\mathsf{A}} \cap I^{\mathsf{B}}$.

8.3 Differentiability

We present a definition of differentiability that picks up from the previous discussion. It requires a set of pre-conditions on the kinds of test fluent that are admissible, which we make explicit here.

Definition 8.1. Suppose for simplicity that $U \subseteq \mathbb{R}$ is a non-trivial open interval. The fluent $x^\star = (\mathbb{R}, U, I, \nu)$ is an *eligible test fluent for differentiability at* $x_0 \in U$ iff it satisfies the following two properties:

- $x^\star[t_0] = x_0 \in U$ for some t_0 straddled by $x^\star$'s indexing interval I; and
- $x^\star$ has a unique flow velocity limit $\dot{x}^\star|_{t_0}$ through t_0.

We can make such a t_0 explicit by saying $x^\star$ is an *eligible test fluent for differentiability at* $x_0 \in U$ *through* t_0. (In principle, there may be more than one such t_0.)

The fluent $x^\star = (\mathbb{R}, U, I, \nu)$ is a *uniform flow velocity test fluent for differentiability at* $x_0 \in U$ iff additionally:

- $x^\star$ has uniform flow velocity on I.

Remark 8.2. Definition 8.1 can be applied to a fluent continuation whenever a suitable restriction can be found.

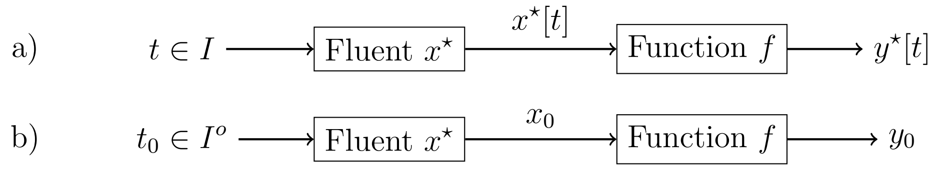

Figure 8.1: The elements of a function differentiable with respect to a test fluent $x^\star$ at $x_0 \in U$ through a time-like instant $t_0 \in I^o$: a) as t varies, $x^\star[t]$ and the induced fluent $y^\star[t] = (f \circ x^\star)[t]$ vary with it; and b) the three spot values t_0, $x_0 = x^\star[t_0]$ and $y_0 = f(x_0)$ are all aligned.

Definition 8.2. Suppose $U \subseteq \mathbb{R}$ is a non-trivial open interval, $x_0 \in U$ and $x^\star$ is an eligible test fluent for differentiability at x_0 through t_0 (ie. $x^\star$ satisfies the conditions in Definition 8.1). The real function $f\colon U \to \mathbb{R}$ is then *differentiable at* $x_0 \in U$ *with respect to fluent* $x^\star$ *through* t_0 iff the induced fluent $y^\star = f \circ x^\star$ has a unique flow velocity limit $\dot{y}^\star|_{t_0}$ through t_0.

The real function $f\colon U \to \mathbb{R}$ is *differentiable at* $x_0 \in U$ iff, for every eligible test fluent $x^\star$ for differentiability at x_0, it is differentiable at $x_0 \in U$ with respect to $x^\star$.

Corollary 8.1. *When* $f\colon U \to \mathbb{R}$ *is differentiable with respect to the eligible test fluent* $x^\star$ *at* x_0 *through* t_0*, then the induced fluent* $y^\star = f \circ x^\star$ *will satisfy the following set of properties, closely paralleling* $x^\star$*:*

- $y^\star$ *has the same indexing interval* I*, which straddles* t_0*;*
- $y^\star$ *has values in* $\mathbb{R}$*;*
- $y^\star[t_0] = y_0 = f(x_0) \in \mathbb{R}$*; and*
- *(from the definition)* $y^\star$ *has a unique flow velocity limit* $\dot{y}^\star|_{t_0}$ *through* t_0*.*

Proof. These follow directly from the definition of an induced fluent $y^\star = f \circ x^\star$ and Definition 8.2. ☐

Example 8.1. The identity function $\mathrm{id}_{\mathbb{R}}(x) = x$ is differentiable at every $x_0 \in \mathbb{R}$ according to Definition 8.2; so we can be assured there's at least one differentiable function.

Example 8.2. For any suitable non-trivial open interval $U \subseteq \mathbb{R}$ and $x_0 \in U$, the function $f(x) = x^2$ is differentiable at x_0 with respect to fluent continuation ψ_1.

Example 8.3. The modulus function $f(x) = |x|$ is everywhere differentiable with respect to the fluent continuation ${\psi_1}^2\,[t] \equiv t^2$, but is not differentiable at $x = 0$ with respect to the fluent continuation ψ_1; ie. it isn't differentiable everywhere for every eligible test fluent.

Example 8.4. The constant function $f(x) = 1$ is everywhere differentiable with respect to any eligible test fluent.

8.4 Sign of a Derivative

The current definition of differentiability requires a set of conditions to be met by *all* eligible test fluents $x^\star$. It prompts two immediate objectives: to come up with a simpler, sufficient requirement; and to evolve a worthwhile differential calculus. Ultimately, I'll show that a function is differentiable with respect to all eligible test fluents if it is differentiable with respect to at least one eligible test fluent with non-zero uniform flow velocity.

In doing this, the reasoning mostly proceeds by comparing flow velocities. Manipulation of the inequalities involved can be simplified by first classifying derivatives by sign.

Definition 8.3. Suppose $U \subseteq \mathbb{R}$ is a non-trivial open interval, $x_0 \in U$, $x^\star$ is an eligible test fluent for differentiability at x_0 through $t_0^\star$, and the real function $f\colon U \to \mathbb{R}$ is differentiable at $x_0 \in U$ with respect to fluent $x^\star$ through $t_0^\star$. Hence the induced fluent $y^\star = f \circ x^\star$ has a unique flow velocity limit $\dot{y}^\star|_{t_0^\star}$, as in Definition 8.2.

We can classify the derivative as follows.

If $\dot{x}^\star|_{t_0^\star} > 0$, then:

- f has *negative derivative at x_0 with respect to $x^\star$ through $t_0^\star$* iff f is differentiable at x_0 with respect to $x^\star$ through $t_0^\star$ and $\dot{y}^\star|_{t_0^\star} < 0$;
- f has *positive derivative at x_0 with respect to $x^\star$ through $t_0^\star$* iff f is differentiable at x_0 with respect to $x^\star$ through $t_0^\star$ and $\dot{y}^\star|_{t_0^\star} > 0$;
- and f has *zero derivative at x_0 with respect to $x^\star$ through $t_0^\star$* iff f is differentiable at x_0 with respect to $x^\star$ through $t_0^\star$ and $\dot{y}^\star|_{t_0^\star} = 0$.

If $\dot{x}^\star|_{t_0^\star} < 0$, then:

- f has *negative derivative at x_0 with respect to $x^\star$ through $t_0^\star$* iff f is differentiable at x_0 with respect to $x^\star$ through $t_0^\star$ and $\dot{y}^\star|_{t_0^\star} > 0$;
- f has *positive derivative at x_0 with respect to $x^\star$ through $t_0^\star$* iff f is differentiable at x_0 with respect to $x^\star$ through $t_0^\star$ and $\dot{y}^\star|_{t_0^\star} < 0$;
- and f has *zero derivative at x_0 with respect to $x^\star$ through $t_0^\star$* iff f is differentiable at x_0 with respect to $x^\star$ through $t_0^\star$ and $\dot{y}^\star|_{t_0^\star} = 0$.

(A negative derivative reverses the sign of $\dot{y}^\star|_{t_0^\star}$, compared to that of $\dot{x}^\star|_{t_0^\star}$; while a positive derivative leaves the sign unchanged.)

Corollary 8.2. *Using the same terminology as Definition 8.3, the interplay between the sign of the derivative and of f is as one would expect.*

- *The function $-f$ has negative derivative at x_0 with respect to $x^\star$ through $t_0^\star$ iff f has positive derivative at x_0 with respect to $x^\star$ through $t_0^\star$.*
- *The function $-f$ has zero derivative at x_0 with respect to $x^\star$ through $t_0^\star$ iff f has zero derivative at x_0 with respect to $x^\star$ through $t_0^\star$.*

Proof. The proof follows from Definition 8.3, noting that $\operatorname{flx}(-f) \circ x^\star = \operatorname{flx} -(f \circ x^\star) = -\operatorname{flx} f \circ x^\star$ through $t_0^\star$. $\square$

8.5 Fundamental Fluxion Identity

This section establishes the fundamental fluxion identity (FFI) that leads to defining the derivative of a function. It proceeds by a number of stages, starting with the special case of all the main contributory entities being positive. For simplicity, for much of this elaboration, one of the two fluents involved has uniform flow velocity (denoted x^{B}), while the other is more general (denoted x^{A}).

8.5.1 FFI for All Entities Positive

The next theorem puts an upper bound on the flow velocity of an induced fluent, when the underlying function is differentiable with respect to a fluent with uniform flow velocity. It simplifies the inequalities arising by requiring that all the entities of interest are positive.

Theorem 8.3. *Suppose that the function* $f\colon U \to \mathbb{R}$*, fluent* $x^{\mathsf{B}} = (\mathbb{R}, U, I^{\mathsf{B}}, \nu^{\mathsf{B}})$ *(thought of as a 'base' fluent) and fluent* $x^{\mathsf{A}} = (\mathbb{R}, U, I^{\mathsf{A}}, \nu^{\mathsf{A}})$ *(notionally an 'arbitrary' fluent) satisfy the following:*

- $U \subseteq \mathbb{R}$ *is a non-trivial open interval;*
- x^{B} *has uniform flow velocity* $c^{\mathsf{B}} > 0$ *on* I^{B}*;*
- $x^{\mathsf{B}}[t_0^{\mathsf{B}}] = x_0 \in U$*, where* $t_0^{\mathsf{B}} \in (I^{\mathsf{B}})^o$*;*
- f *has positive derivative with respect to* x^{B} *through* t_0^{B}*; so the induced fluent* $y^{\mathsf{B}} = f \circ x^{\mathsf{B}}$ *has positive unique flow velocity limit* $\dot{y}_0^{\mathsf{B}} \equiv \dot{y}^{\mathsf{B}}|_{t_0^{\mathsf{B}}} > 0$ *through* t_0^{B}*;*
- $x^{\mathsf{A}}[t_0^{\mathsf{A}}] = x_0$ *for some* $t_0^{\mathsf{A}} \in (I^{\mathsf{A}})^o$*;*
- *and* x^{A} *has bounded flow velocity through* t_0^{A}*, where* $x^{\mathsf{A}} \gtrdot 0$ *through* t_0^{A}*.*

Now suppose $v^{\mathsf{B}} \in \mathbb{R}$ *and* $u^{\mathsf{A}} \in \mathbb{R}$ *satisfy:*

- $y^{\mathsf{B}} \lessdot v^{\mathsf{B}}\psi_1$ *through* t_0^{B}*;*
- *and* $x^{\mathsf{A}} \lessdot u^{\mathsf{A}}\psi_1$ *through* t_0^{A}*.*

Write $y^{\mathsf{A}} \equiv f \circ x^{\mathsf{A}}$*. Then:*

- $0 \lessdot c^{\mathsf{B}} y^{\mathsf{A}} \lessdot v^{\mathsf{B}} u^{\mathsf{A}} \psi_1$ *through* t_0^{A}*;*
- *so* $0 \leq c^{\mathsf{B}} \acute{y}_0^{\mathsf{A}} \leq v^{\mathsf{B}} u^{\mathsf{A}}$*, for the least upper flow velocity*$^{(Def\ 6.2)}$ $\acute{y}_0^{\mathsf{A}} \equiv (\operatorname{lufv} y^{\mathsf{A}})|_{t_0^{\mathsf{A}}}$*.*

Proof. The proof can be divided into several steps.

1. For each closed sub-interval $J^{\mathsf{B}} \subseteq I^{\mathsf{B}}$, fluent x^{B} satisfies:

   $$\Delta x^{\mathsf{B}}|_{J^{\mathsf{B}}} = c^{\mathsf{B}} \Delta J^{\mathsf{B}}.$$

2. By construction of the flow velocity limit (and Theorem 6.2), there is a closed sub-interval $I^{\mathsf{B}}_{\mathsf{v}} \subseteq I^{\mathsf{B}}$ straddling t^{B}_0 where $y^{\mathsf{B}} \lessdot v^{\mathsf{B}}\psi_1$ on $I^{\mathsf{B}}_{\mathsf{v}}$ through t^{B}_0.

   Because $0 < \dot{y}^{\mathsf{B}}_0$, it follows that $0 \lessdot y^{\mathsf{B}}$ through t^{B}_0; and there is a closed sub-interval $I^{\mathsf{B}}_0 \subseteq I^{\mathsf{B}}$ straddling t^{B}_0 where $0 \lessdot y^{\mathsf{B}}$ on I^{B}_0 through t^{B}_0.

   Form the closed sub-interval $I^{\mathsf{B}}_* \equiv I^{\mathsf{B}}_{\mathsf{v}} \cap I^{\mathsf{B}}_0 \subseteq I^{\mathsf{B}}$, which again straddles t^{B}_0.

   Then $0 \lessdot y^{\mathsf{B}} \lessdot v^{\mathsf{B}}\psi_1$ on I^{B}_* through t^{B}_0. Hence, for all non-trivial closed sub-intervals $J^{\mathsf{B}} \subseteq I^{\mathsf{B}}_*$ containing t^{B}_0:

   $$0 < \Delta y^{\mathsf{B}}|_{J^{\mathsf{B}}} < v^{\mathsf{B}} \Delta J^{\mathsf{B}}.$$

3. By a similar argument, there is a closed sub-interval $I^{\mathsf{A}}_0 \subseteq I^{\mathsf{A}}$ straddling t^{A}_0 where $0 \lessdot x^{\mathsf{A}}$ on I^{A}_0 through t^{A}_0.

   Also, there is a closed sub-interval $I^{\mathsf{A}}_{\mathsf{u}} \subseteq I^{\mathsf{A}}$ straddling t^{A}_0 where $x^{\mathsf{A}} \lessdot u^{\mathsf{A}}\psi_1$ on $I^{\mathsf{A}}_{\mathsf{u}}$ through t^{A}_0.

   Fluent x^{A} has bounded flow velocity through t^{A}_0, so by Theorem 6.13 it is within-bounds continuous through t^{A}_0 and by Corollary 4.3 there is an interval $H^{\mathsf{A}} \subseteq I^{\mathsf{A}}$ straddling t^{A}_0 for which $\nu^{\mathsf{A}}(H^{\mathsf{A}}) \subseteq \nu^{\mathsf{B}}(I^{\mathsf{B}}_*)$.

   Now form the closed sub-interval $I^{\mathsf{A}}_* \equiv I^{\mathsf{A}}_0 \cap I^{\mathsf{A}}_{\mathsf{u}} \cap H^{\mathsf{A}}$, which straddles t^{A}_0.

   Also, $0 \lessdot x^{\mathsf{A}} \lessdot u^{\mathsf{A}}\psi_1$ on I^{A}_* through t^{A}_0. Hence, for all non-trivial closed sub-intervals $J^{\mathsf{A}} \subseteq I^{\mathsf{A}}_*$ containing t^{A}_0:

   $$0 < \Delta x^{\mathsf{A}}|_{J^{\mathsf{A}}} < u^{\mathsf{A}} \Delta J^{\mathsf{A}}.$$

4. By construction, $\nu^{\mathsf{A}}(I^{\mathsf{A}}_*) \subseteq \nu^{\mathsf{B}}(I^{\mathsf{B}}_*)$. For any non-trivial closed sub-interval $J^{\mathsf{A}} = [t^{\mathsf{A}}_1, t^{\mathsf{A}}_2] \subseteq I^{\mathsf{A}}_*$ containing t^{A}_0, define:

   $$\begin{aligned} x_1 &\equiv x^{\mathsf{A}}[t^{\mathsf{A}}_1] \in \nu^{\mathsf{B}}(I^{\mathsf{B}}_*), \\ x_2 &\equiv x^{\mathsf{A}}[t^{\mathsf{A}}_2] \in \nu^{\mathsf{B}}(I^{\mathsf{B}}_*). \end{aligned}$$

   Put $J^{\mathsf{A}}_1 = [t^{\mathsf{A}}_1, t^{\mathsf{A}}_0]$ and $J^{\mathsf{A}}_2 = [t^{\mathsf{A}}_0, t^{\mathsf{A}}_2]$. Then $0 < \Delta x^{\mathsf{A}}|_{J^{\mathsf{A}}_1}$ and $0 < \Delta x^{\mathsf{A}}|_{J^{\mathsf{A}}_2}$, because $0 \lessdot x^{\mathsf{A}}$ on I^{A}_* through t^{A}_0; so $x_1 < x_0 < x_2$.

   The values x_1 and x_2 form a non-trivial sub-interval $[x_1, x_2] \subseteq \nu^{\mathsf{B}}(I^{\mathsf{B}}_*)$. This has a well-defined non-trivial pre-image $J^{\mathsf{B}} \equiv [t^{\mathsf{B}}_1, t^{\mathsf{B}}_2] \subseteq I^{\mathsf{B}}_*$ containing t^{B}_0, where $x^{\mathsf{B}}[t^{\mathsf{B}}_1] = x_1$ and $x^{\mathsf{B}}[t^{\mathsf{B}}_2] = x_2$, because x^{B} has positive uniform flow velocity on I^{B}. Hence:

   $$\begin{aligned} \Delta x^{\mathsf{A}}|_{J^{\mathsf{A}}} &= x_2 - x_1 = \Delta x^{\mathsf{B}}|_{J^{\mathsf{B}}} > 0, \\ \Delta y^{\mathsf{A}}|_{J^{\mathsf{A}}} &= f(x_2) - f(x_1) = \Delta y^{\mathsf{B}}|_{J^{\mathsf{B}}} > 0. \end{aligned}$$

5. Combining the above relations (and noting that all quantities are positive, except 0 itself):

$$\begin{aligned} 0 < c^{\mathsf{B}}\Delta y^{\mathsf{A}}|_{J^{\mathsf{A}}} &= c^{\mathsf{B}}\Delta y^{\mathsf{B}}|_{J^{\mathsf{B}}} \\ &< c^{\mathsf{B}}v^{\mathsf{B}}\Delta J^{\mathsf{B}} \\ &= v^{\mathsf{B}}\Delta x^{\mathsf{B}}|_{J^{\mathsf{B}}} \\ &= v^{\mathsf{B}}\Delta x^{\mathsf{A}}|_{J^{\mathsf{A}}} \\ &< v^{\mathsf{B}}u^{\mathsf{A}}\Delta J^{\mathsf{A}}, \end{aligned}$$

   for all non-trivial closed sub-intervals $J^{\mathsf{A}} \subseteq I_*^{\mathsf{A}}$ containing t_0^{A} and corresponding $J^{\mathsf{B}} \subseteq I_*^{\mathsf{B}}$ containing t_0^{B}.

   As required, we therefore have:

$$0 \lessdot c^{\mathsf{B}}y^{\mathsf{A}} \lessdot v^{\mathsf{B}}u^{\mathsf{A}}\psi_1 \quad \text{through } t_0^{\mathsf{A}},$$

6. Finally, apply Theorem 6.2 to establish that:

$$0 \leq c^{\mathsf{B}}\acute{y}_0^{\mathsf{A}} \leq v^{\mathsf{B}}u^{\mathsf{A}}.$$

This completes the proof. □

The next theorem contains a dual form with regard to fluent x^{A}. The main preconditions are again those of Theorem 8.3.

Theorem 8.4. *Suppose that the function* $f\colon U \to \mathbb{R}$, *fluent* $x^{\mathsf{B}} = (\mathbb{R}, U, I^{\mathsf{B}}, \nu^{\mathsf{B}})$ *and fluent* $x^{\mathsf{A}} = (\mathbb{R}, U, I^{\mathsf{A}}, \nu^{\mathsf{A}})$ *satisfy the following:*

- $U \subseteq \mathbb{R}$ *is a non-trivial open interval;*
- x^{B} *has uniform flow velocity* $c^{\mathsf{B}} > 0$ *on* I^{B}*;*
- $x^{\mathsf{B}}[t_0^{\mathsf{B}}] = x_0 \in U$*, where* $t_0^{\mathsf{B}} \in (I^{\mathsf{B}})^o$*;*
- f *has positive derivative with respect to* x^{B} *through* t_0^{B}*; so the induced fluent* $y^{\mathsf{B}} = f \circ x^{\mathsf{B}}$ *has positive unique flow velocity limit* $\dot{y}_0^{\mathsf{B}} \equiv \dot{y}^{\mathsf{B}}|_{t_0^{\mathsf{B}}} > 0$ *through* t_0^{B}*;*
- $x^{\mathsf{A}}[t_0^{\mathsf{A}}] = x_0$ *for some* $t_0^{\mathsf{A}} \in (I^{\mathsf{A}})^o$*;*
- *and* x^{A} *has bounded flow velocity through* t_0^{A}*, where* $x^{\mathsf{A}} \gtrdot 0$ *through* t_0^{A}*.*

Now suppose $v^{\mathsf{B}} \in \mathbb{R}$ *and* $u^{\mathsf{A}} \in \mathbb{R}$ *satisfy* $v^{\mathsf{B}} > 0$ *and* $u^{\mathsf{A}} > 0$*, and:*

- $v^{\mathsf{B}}\psi_1 \lessdot y^{\mathsf{B}}$ *through* t_0^{B}*;*

- *and* $u^{\mathsf{A}}\psi_1 \lessdot x^{\mathsf{A}}$ *through* t_0^{A}.

Write $y^{\mathsf{A}} \equiv f \circ x^{\mathsf{A}}$. *Then:*

- $0 \lessdot v^{\mathsf{B}}u^{\mathsf{A}}\psi_1 \lessdot c^{\mathsf{B}}y^{\mathsf{A}}$ *through* t_0^{A}*;*
- *so* $0 < v^{\mathsf{B}}u^{\mathsf{A}} \leq c^{\mathsf{B}}\grave{y}_0^{\mathsf{A}}$, *for the greatest lower flow velocity*$^{(\textit{Def 6.2})}$ $\grave{y}_0^{\mathsf{A}} \equiv (\operatorname{glfv} y^{\mathsf{A}})|_{t_0^{\mathsf{A}}}$.

Proof. The proof is a variant of the proof of Theorem 8.3, so some parts are skimmed over lightly. Note that by Lemma 5.4, if $v^{\mathsf{B}} > 0$ and $u^{\mathsf{A}} > 0$, then $0 \lessdot v^{\mathsf{B}}\psi_1$ and $0 \lessdot u^{\mathsf{A}}\psi_1$.

1. For each closed sub-interval $J^{\mathsf{B}} \subseteq I^{\mathsf{B}}$, fluent x^{B} satisfies:
$$\Delta x^{\mathsf{B}}|_{J^{\mathsf{B}}} = c^{\mathsf{B}}\Delta J^{\mathsf{B}}.$$
2. There is a closed sub-interval $I_*^{\mathsf{B}} \subseteq I^{\mathsf{B}}$ straddling t_0^{B}, such that $0 \lessdot v^{\mathsf{B}}\psi_1 \lessdot y^{\mathsf{B}}$ on I_*^{B} through t_0^{B}. Hence, for all non-trivial closed sub-intervals $J^{\mathsf{B}} \subseteq I_*^{\mathsf{B}}$ containing t_0^{B}:
$$0 < v^{\mathsf{B}}\Delta J^{\mathsf{B}} < \Delta y^{\mathsf{B}}|_{J^{\mathsf{B}}}.$$
3. Construct closed sub-interval $I_*^{\mathsf{A}} \subseteq I^{\mathsf{A}}$ straddling t_0^{A}, such that:
$$\begin{aligned} &\nu^{\mathsf{A}}(I_*^{\mathsf{A}}) \subseteq \nu^{\mathsf{B}}(I_*^{\mathsf{B}}); \\ &\quad 0 \lessdot u^{\mathsf{A}}\psi_1 \lessdot x^{\mathsf{A}} \qquad \text{on } I_*^{\mathsf{A}} \text{ through } t_0^{\mathsf{A}}. \end{aligned}$$
Hence, for all non-trivial closed sub-intervals $J^{\mathsf{A}} \subseteq I_*^{\mathsf{A}}$ containing t_0^{A}:
$$0 < u^{\mathsf{A}}\Delta J^{\mathsf{A}} < \Delta x^{\mathsf{A}}|_{J^{\mathsf{A}}}.$$
4. For any non-trivial closed sub-interval $J^{\mathsf{A}} = [t_1^{\mathsf{A}}, t_2^{\mathsf{A}}] \subseteq I_*^{\mathsf{A}}$ containing t_0^{A}, define:
$$\begin{aligned} x_1 &\equiv x^{\mathsf{A}}[t_1^{\mathsf{A}}] \in \nu^{\mathsf{B}}(I_*^{\mathsf{B}}), \\ x_2 &\equiv x^{\mathsf{A}}[t_2^{\mathsf{A}}] \in \nu^{\mathsf{B}}(I_*^{\mathsf{B}}). \end{aligned}$$
Then $x_1 < x_0 < x_2$, because $0 \lessdot x^{\mathsf{A}}$ on I_*^{A} through t_0^{A}.

   The non-trivial sub-interval $[x_1, x_2] \subseteq \nu^{\mathsf{B}}(I_*^{\mathsf{B}})$ has a well-defined non-trivial pre-image $J^{\mathsf{B}} \equiv [t_1^{\mathsf{B}}, t_2^{\mathsf{B}}] \subseteq I_*^{\mathsf{B}}$ containing t_0^{B}. Hence:
$$\begin{aligned} \Delta x^{\mathsf{A}}|_{J^{\mathsf{A}}} &= x_2 - x_1 = \Delta x^{\mathsf{B}}|_{J^{\mathsf{B}}} > 0, \\ \Delta y^{\mathsf{A}}|_{J^{\mathsf{A}}} &= f(x_2) - f(x_1) = \Delta y^{\mathsf{B}}|_{J^{\mathsf{B}}} > 0. \end{aligned}$$

5. Combining the above relations (and noting that all quantities are positive, except 0 itself):

$$\begin{aligned} 0 < v^{\mathsf{B}} u^{\mathsf{A}} \Delta J^{\mathsf{A}} &< v^{\mathsf{B}} \Delta x^{\mathsf{A}}|_{J^{\mathsf{A}}} \\ &= v^{\mathsf{B}} \Delta x^{\mathsf{B}}|_{J^{\mathsf{B}}} \\ &= c^{\mathsf{B}} v^{\mathsf{B}} \Delta J^{\mathsf{B}} \\ &< c^{\mathsf{B}} \Delta y^{\mathsf{B}}|_{J^{\mathsf{B}}} \\ &= c^{\mathsf{B}} \Delta y^{\mathsf{A}}|_{J^{\mathsf{A}}}, \end{aligned}$$

   for all non-trivial closed sub-intervals $J^{\mathsf{A}} \subseteq I^{\mathsf{A}}_*$ containing t^{A}_0 and corresponding $J^{\mathsf{B}} \subseteq I^{\mathsf{B}}_*$ containing t^{B}_0.

   Therefore, as required:

$$0 \lessdot v^{\mathsf{B}} u^{\mathsf{A}} \psi_1 \lessdot c^{\mathsf{B}} y^{\mathsf{A}} \quad \text{through } t^{\mathsf{A}}_0.$$

6. Finally, apply Theorem 6.2 to obtain:

$$0 < v^{\mathsf{B}} u^{\mathsf{A}} \leq c^{\mathsf{B}} \grave{y}^{\mathsf{A}}_0.$$

The last two results complete the proof. □

The next theorem borrows its preconditions from Theorem 8.3 and Theorem 8.4. It produces a first form of the fundamental fluxion identity.

Theorem 8.5. *Suppose that the function* $f\colon U \to \mathbb{R}$, *fluent* $x^{\mathsf{B}} = (\mathbb{R}, U, I^{\mathsf{B}}, \nu^{\mathsf{B}})$ *and fluent* $x^{\mathsf{A}} = (\mathbb{R}, U, I^{\mathsf{A}}, \nu^{\mathsf{A}})$ *satisfy the following:*

- $U \subseteq \mathbb{R}$ *is a non-trivial open interval;*
- x^{B} *has uniform flow velocity* $c^{\mathsf{B}} > 0$ *on* I^{B}*;*
- $x^{\mathsf{B}}[t^{\mathsf{B}}_0] = x_0 \in U$*, where* $t^{\mathsf{B}}_0 \in (I^{\mathsf{B}})^o$*;*
- f *has positive derivative with respect to* x^{B} *through* t^{B}_0*; so the induced fluent* $y^{\mathsf{B}} = f \circ x^{\mathsf{B}}$ *has positive unique flow velocity limit* $\dot{y}^{\mathsf{B}}_0 \equiv \dot{y}^{\mathsf{B}}|_{t^{\mathsf{B}}_0} > 0$ *through* t^{B}_0*;*
- $x^{\mathsf{A}}[t^{\mathsf{A}}_0] = x_0$ *for some* $t^{\mathsf{A}}_0 \in (I^{\mathsf{A}})^o$*;*
- *and* x^{A} *has bounded flow velocity through* t^{A}_0*, where* $x^{\mathsf{A}} \gtrdot 0$ *through* t^{A}_0*.*

Construct the induced fluent $y^{\mathsf{A}} \equiv f \circ x^{\mathsf{A}}$*, which has bounded flow velocity by the previous two theorems. Then* y^{A} *satisfies the following.*

- *(Upper flow velocity bound) If* $\acute{x}_0^{\mathsf{A}} > 0$*, then* $0 \leq c^{\mathsf{B}}\, \acute{y}_0^{\mathsf{A}} \leq \acute{x}_0^{\mathsf{A}}\, \dot{y}_0^{\mathsf{B}}$*, where* $\acute{x}_0^{\mathsf{A}} \equiv (\mathrm{lufv}\, x^{\mathsf{A}})|_{t_0^{\mathsf{A}}}$ *and* $\acute{y}_0^{\mathsf{A}} \equiv (\mathrm{lufv}\, y^{\mathsf{A}})|_{t_0^{\mathsf{A}}}$*.*
- *(Lower flow velocity bound) If* $\grave{x}_0^{\mathsf{A}} > 0$*, then* $0 \leq \grave{x}_0^{\mathsf{A}}\, \dot{y}_0^{\mathsf{B}} \leq c^{\mathsf{B}}\, \grave{y}_0^{\mathsf{A}}$*, where* $\grave{x}_0^{\mathsf{A}} \equiv (\mathrm{glfv}\, x^{\mathsf{A}})|_{t_0^{\mathsf{A}}}$ *and* $\grave{y}_0^{\mathsf{A}} \equiv (\mathrm{glfv}\, y^{\mathsf{A}})|_{t_0^{\mathsf{A}}}$*.*
- *(Prototype fluxion identity) If the fluent* x^{A} *has unique flow velocity limit* $\dot{x}_0^{\mathsf{A}} \equiv \dot{x}^{\mathsf{A}}|_{t_0^{\mathsf{A}}}$ *through* t_0^{A} *and* $\dot{x}_0^{\mathsf{A}} > 0$*, then the induced fluent* y^{A} *has unique flow velocity limit through* t_0^{A}*, namely* $\dot{y}_0^{\mathsf{A}} \equiv \dot{y}^{\mathsf{A}}|_{t_0^{\mathsf{A}}}$*; and* $c^{\mathsf{B}}\, \dot{y}_0^{\mathsf{A}} = \dot{x}_0^{\mathsf{A}}\, \dot{y}_0^{\mathsf{B}}$*.*

Proof. We prove the three bulleted results in turn. The second result is only briefly sketched.

For the first bulleted result (Upper flow velocity bound), borrow the notation and conclusion of Theorem 8.3; and apply the second assertion of Lemma 1.1 to $\acute{x}_0^{\mathsf{A}} < u^{\mathsf{A}}$ and $\dot{y}_0^{\mathsf{B}} < v^{\mathsf{B}}$ in turn.

Note that if $\acute{x}_0^{\mathsf{A}} < u^{\mathsf{A}}$ and $\dot{y}_0^{\mathsf{B}} < v^{\mathsf{B}}$, then $x_0^{\mathsf{A}} \lessdot u^{\mathsf{A}}\psi_1$ through t_0^{A} and $y_0^{\mathsf{B}} \lessdot v^{\mathsf{B}}\psi_1$ through t_0^{B}. Hence Theorem 8.3 shows that given $v^{\mathsf{B}} > \dot{y}_0^{\mathsf{B}} > 0$, and $u^{\mathsf{A}} > \acute{x}_0^{\mathsf{A}}$, then:

$$\frac{c^{\mathsf{B}}}{v^{\mathsf{B}}}\, \acute{y}_0^{\mathsf{A}} \leq u^{\mathsf{A}}.$$

From Lemma 1.1, this implies:

$$\frac{c^{\mathsf{B}}}{v^{\mathsf{B}}}\, \acute{y}_0^{\mathsf{A}} \leq \acute{x}_0^{\mathsf{A}}.$$

Noting that $\acute{x}_0^{\mathsf{A}} > 0$ and rearranging implies that, for all $v^{\mathsf{B}} > \dot{y}_0^{\mathsf{B}} > 0$:

$$c^{\mathsf{B}}\, \frac{\acute{y}_0^{\mathsf{A}}}{\acute{x}_0^{\mathsf{A}}} \leq v^{\mathsf{B}}.$$

Applying Lemma 1.1 again and rearranging, then:

$$0 \leq c^{\mathsf{B}}\, \acute{y}_0^{\mathsf{A}} \leq \acute{x}_0^{\mathsf{A}}\, \dot{y}_0^{\mathsf{B}},$$

where it's easy to check that the expressions are non-negative.

The second bulleted result (Lower flow velocity bound) is proven similarly. In this case, take the the notation and conclusion of Theorem 8.4; and apply a variant of the first assertion of Lemma 1.1 to $u^{\mathsf{A}} < \grave{x}_0^{\mathsf{A}}$ and $v^{\mathsf{B}} < \dot{y}_0^{\mathsf{B}}$ in turn, accounting for the inequalities $u_0^{\mathsf{A}} > 0$ and $v^{\mathsf{B}} > 0$.

For the final result (Prototype fluxion identity), we can rewrite the previous relations with $\acute{x}_0^{\mathsf{A}} = \grave{x}_0^{\mathsf{A}} = \dot{x}_0^{\mathsf{A}}$, yielding two inequalities:

$$\begin{gathered} c^{\mathsf{B}}\, \acute{y}_0^{\mathsf{A}} \leq \dot{x}_0^{\mathsf{A}}\, \dot{y}_0^{\mathsf{B}}, \\ \dot{x}_0^{\mathsf{A}}\, \dot{y}_0^{\mathsf{B}} \leq c^{\mathsf{B}}\, \grave{y}_0^{\mathsf{A}}. \end{gathered}$$

Combine them to get $c^{\mathsf{B}}\,\acute{y}^{\mathsf{A}}_0 \le \dot{x}^{\mathsf{A}}_0\,\dot{y}^{\mathsf{B}}_0 \le c^{\mathsf{B}}\,\grave{y}^{\mathsf{A}}_0$. But $\grave{y}^{\mathsf{A}}_0 \le \acute{y}^{\mathsf{A}}_0$ and c^{B} is positive, so $\grave{y}^{\mathsf{A}}_0 = \acute{y}^{\mathsf{A}}_0 = \dot{y}^{\mathsf{A}}_0$; and:

$$c^{\mathsf{B}}\,\dot{y}^{\mathsf{A}}_0 = \dot{x}^{\mathsf{A}}_0\,\dot{y}^{\mathsf{B}}_0.$$

The proof is now complete. $\square$

8.5.2 FFI for All Entities Non-zero

We've now established a fundamental fluxion identity under special conditions, namely those of Theorem 8.5, where the key ingredients were positive. While positivity was useful in manipulating inequalities, you might suspect that equations are less susceptible to sign changes than inequalities would be. This is confirmed by the next theorem. Although we've met the fundamental fluxion identity several times already, a formal definition helps here.

The definition is supposed to reflect the fact that the fundamental fluxion identity comes in various flavours, depending on the nature of the function and the fluents involved.

Definition 8.4. Suppose we have a function $f\colon U \to \mathbb{R}$, where $U \subseteq \mathbb{R}$ is a non-trivial open interval. Suppose also that the fluents $x^{\mathsf{B}} = (\mathbb{R}, U, I^{\mathsf{B}}, \nu^{\mathsf{B}})$ and $x^{\mathsf{A}} = (\mathbb{R}, U, I^{\mathsf{A}}, \nu^{\mathsf{A}})$ are eligible test fluents for differentiability at $x_0 \in U$ through $t^{\mathsf{A}}_0 \in (I^{\mathsf{A}})^o$ and $t^{\mathsf{B}}_0 \in (I^{\mathsf{B}})^o$ respectively$^{(\text{Def } 8.1)}$.

Then a *fundamental fluxion identity (FFI)* takes the form

$$\dot{x}^{\mathsf{B}}\,\dot{y}^{\mathsf{A}} = \dot{x}^{\mathsf{A}}\,\dot{y}^{\mathsf{B}} \quad \text{through } t^{\mathsf{A}}_0 \text{ and } t^{\mathsf{B}}_0,$$

where $y^{\mathsf{A}} = f \circ x^{\mathsf{A}}$ and $y^{\mathsf{B}} = f \circ x^{\mathsf{B}}$.

Equivalently:

$$\dot{x}^{\mathsf{B}}\,\operatorname{flx} f \circ x^{\mathsf{A}} = \dot{x}^{\mathsf{A}}\,\operatorname{flx} f \circ x^{\mathsf{B}} \quad \text{through } t^{\mathsf{A}}_0 \text{ and } t^{\mathsf{B}}_0.$$

When x^{B} has uniform flow velocity $c^{\mathsf{B}} > 0$, it is a uniform flow velocity test fluent for differentiability, and the corresponding FFI may be written:

$$c^{\mathsf{B}}\,\dot{y}^{\mathsf{A}} = \dot{x}^{\mathsf{A}}\,\dot{y}^{\mathsf{B}} \quad \text{through } t^{\mathsf{A}}_0 \text{ and } t^{\mathsf{B}}_0.$$

Theorem 8.6. *The notation is the same as Definition 8.4.*

1. *If the FFI holds whenever f has positive derivative with respect to x^{B}, then it holds when f has negative derivative with respect to x^{B}.*

2. *If the FFI holds whenever $\dot{x}^{\mathsf{B}}|_{t_0} > 0$, then it holds when $\dot{x}^{\mathsf{B}}|_{t_0} < 0$.*

3. *If the FFI holds whenever $\dot{x}^{\mathsf{A}}|_{t_0} > 0$, then it holds when $\dot{x}^{\mathsf{A}}|_{t_0} < 0$.*

Proof. Take each result in turn.

1. Assume the FFI holds for all f when f has positive derivative with respect to x^{B}. Suppose $g\colon U \to \mathbb{R}$ has negative derivative with respect to x^{B}. We want to show that the FFI holds for g.

   Corollary 8.2 implies $-g$ has positive derivative with respect to x^{B}. Hence:

   $$\dot{x}^{\mathsf{B}}\ \mathrm{flx}\,(-g) \circ x^{\mathsf{A}} = \dot{x}^{\mathsf{A}}\ \mathrm{flx}\,(-g) \circ x^{\mathsf{B}} \quad \text{through } t_0^{\mathsf{A}} \text{ and } t_0^{\mathsf{B}}.$$

   But $\mathrm{flx}\,(-g) \circ x^{\mathsf{A}} = \mathrm{flx}\ -(g \circ x^{\mathsf{A}}) = -\,\mathrm{flx}\ g \circ x^{\mathsf{A}}$; and similarly $\mathrm{flx}\,(-g) \circ x^{\mathsf{B}} = -\,\mathrm{flx}\ g \circ x^{\mathsf{B}}$. Hence:

   $$-\dot{x}^{\mathsf{B}}\ \mathrm{flx}\ g \circ x^{\mathsf{A}} = -\dot{x}^{\mathsf{A}}\ \mathrm{flx}\ g \circ x^{\mathsf{B}} \quad \text{through } t_0^{\mathsf{A}} \text{ and } t_0^{\mathsf{B}}.$$

   Multiplying both sides by -1 yields the FFI for g, as required.

2. Assume the FFI holds for all eligible test fluents x^{B} for which $\dot{x}^{\mathsf{B}}|_{t_0} > 0$. Now take an eligible test fluent x^{C} with $\dot{x}^{\mathsf{C}}|_{t_0} < 0$, where $x^{\mathsf{C}}[t_0^{\mathsf{C}}] = x_0$. We wish to show that the FFI holds for x^{A} and x^{C}.

   The duality result of Theorem 7.1 implies that $(\mathrm{flx}\,\mathrm{opp}\,x^{\mathsf{C}})|_{-t_0} = -(\mathrm{flx}\,x^{\mathsf{C}})|_{t_0} > 0$. Hence:

   $$(\mathrm{flx}\,\mathrm{opp}\,x^{\mathsf{C}})\,(\mathrm{flx}\,f \circ x^{\mathsf{A}}) = \dot{x}^{\mathsf{A}}\ \mathrm{flx}\,f \circ \mathrm{opp}\,x^{\mathsf{C}} \quad \text{through } t_0^{\mathsf{A}} \text{ and } -t_0^{\mathsf{C}}.$$

   But $(\mathrm{flx}\,f \circ \mathrm{opp}\,x^{\mathsf{C}})|_{-t_0} = (\mathrm{flx}\,\mathrm{opp}\,f \circ x^{\mathsf{C}})|_{-t_0} = -(\mathrm{flx}\,f \circ x^{\mathsf{C}})|_{t_0}$. A similar argument as before yields the FFI for x^{C}, as required.

3. Assume the FFI holds for all eligible test fluents x^{A} for which $\dot{x}^{\mathsf{A}}|_{t_0} > 0$. Now take an eligible test fluent x^{D} with $\dot{x}^{\mathsf{D}}|_{t_0} < 0$, where $x^{\mathsf{D}}[t_0^{\mathsf{D}}] = x_0$. We can show that the FFI holds for x^{A} and x^{D} by symmetry or by repeating the same argument as before.

This completes the proof. □

A corollary summarises the position so far, which is a generalisation of Theorem 8.5.

Corollary 8.7. *Suppose that the function* $f\colon U \to \mathbb{R}$*, fluent* $x^{\mathsf{B}} = (\mathbb{R}, U, I^{\mathsf{B}}, \nu^{\mathsf{B}})$ *and fluent* $x^{\mathsf{A}} = (\mathbb{R}, U, I^{\mathsf{A}}, \nu^{\mathsf{A}})$ *satisfy the following:*

- $U \subseteq \mathbb{R}$ *is a non-trivial open interval;*
- x^{B} *has uniform flow velocity* $c^{\mathsf{B}} \neq 0$ *on* I^{B}*;*

- $x^{\mathsf{B}}[t_0^{\mathsf{B}}] = x_0 \in U$*, where* $t_0^{\mathsf{B}} \in (I^{\mathsf{B}})^o$*;*
- f *has positive or negative derivative with respect to* x^{B} *through* t_0^{B}*; so the induced fluent* $y^{\mathsf{B}} = f \circ x^{\mathsf{B}}$ *has positive or negative unique flow velocity limit* $\dot{y}_0^{\mathsf{B}} \equiv \dot{y}^{\mathsf{B}}|_{t_0^{\mathsf{B}}}$ *through* t_0^{B}*;*
- $x^{\mathsf{A}}[t_0^{\mathsf{A}}] = x_0$ *for some* $t_0^{\mathsf{A}} \in (I^{\mathsf{A}})^o$*;*
- *and* x^{A} *has non-zero unique flow velocity limit* $\dot{x}_0^{\mathsf{A}} \equiv \dot{x}^{\mathsf{A}}|_{t_0^{\mathsf{A}}} \neq 0$ *through* t_0^{A}*.*

Construct the induced fluent $y^{\mathsf{A}} \equiv f \circ x^{\mathsf{A}}$*. Then* y^{A} *has unique flow velocity limit* $\dot{y}_0^{\mathsf{A}} \equiv \dot{y}^{\mathsf{A}}|_{t_0^{\mathsf{A}}}$ *through* t_0^{A}*, and satisfies:*

$$c^{\mathsf{B}}\, \dot{y}_0^{\mathsf{A}} = \dot{x}_0^{\mathsf{A}}\, \dot{y}_0^{\mathsf{B}}.$$

Proof. Theorem 8.5 affirms the case of all entities positive. Any mix of positive and negative entities can be proven by applying Theorem 8.6 repeatedly to Theorem 8.5 as necessary. □

8.5.3 FFI for Some Entities Zero

We've not yet considered any zero cases. When it has zero uniform flow velocity $c^{\mathsf{B}} = 0$, fluent x^{B} will be constant; and hence so will the induced fluent $y^{\mathsf{B}} = f \circ x^{\mathsf{B}}$. Therefore y^{B} will have zero unique flow velocity limit $\dot{y}^{\mathsf{B}} = 0$ through any $t_0^{\mathsf{B}} \in (I^{\mathsf{B}})^o$. With obvious notation, the fluents involved will trivially satisfy the fluxion identity $c^{\mathsf{B}}\, \dot{y}_0^{\mathsf{A}} = \dot{x}_0^{\mathsf{A}}\, \dot{y}_0^{\mathsf{B}}$, which reduces to $0 = 0$.

The next theorem shows that there is a similar result when x^{A} has zero unique flow velocity limit.

Theorem 8.8. *Suppose that the function* $f\colon U \to \mathbb{R}$*, fluent* $x^{\mathsf{B}} = (\mathbb{R}, U, I^{\mathsf{B}}, \nu^{\mathsf{B}})$ *and fluent* $x^{\mathsf{A}} = (\mathbb{R}, U, I^{\mathsf{A}}, \nu^{\mathsf{A}})$ *satisfy the following:*

- $U \subseteq \mathbb{R}$ *is a non-trivial open interval;*
- x^{B} *has uniform flow velocity* $c^{\mathsf{B}} \neq 0$ *on* I^{B}*;*
- $x^{\mathsf{B}}[t_0^{\mathsf{B}}] = x_0 \in U$*, where* $t_0^{\mathsf{B}} \in (I^{\mathsf{B}})^o$*;*
- f *is differentiable at* x_0 *with respect to* x^{B} *through* t_0^{B}*; so the induced fluent* $y^{\mathsf{B}} = f \circ x^{\mathsf{B}}$ *has unique flow velocity limit* $\dot{y}_0^{\mathsf{B}} \equiv \dot{y}^{\mathsf{B}}|_{t_0^{\mathsf{B}}}$ *through* t_0^{B}*;*
- $x^{\mathsf{A}}[t_0^{\mathsf{A}}] = x_0$ *for some* $t_0^{\mathsf{A}} \in (I^{\mathsf{A}})^o$*;*
- *and* x^{A} *has zero unique flow velocity limit* $\dot{x}_0^{\mathsf{A}} \equiv \dot{x}^{\mathsf{A}}|_{t_0^{\mathsf{A}}} = 0$ *through* t_0^{A}*.*

Write $y^{\mathsf{A}} \equiv f \circ x^{\mathsf{A}}$*. Then* y^{A} *has a zero unique flow velocity limit* $\dot{y}^{\mathsf{A}}_0 \equiv \dot{y}^{\mathsf{A}}|_{t^{\mathsf{A}}_0} = 0$ *through* t_0*; and trivially:*

$$c^{\mathsf{B}}\,\dot{y}^{\mathsf{A}}_0 = \dot{x}^{\mathsf{A}}_0\,\dot{y}^{\mathsf{B}}_0.$$

Proof. For each closed sub-interval $J^{\mathsf{B}} \subseteq I^{\mathsf{B}}$, fluent x^{B} satisfies:

$$\Delta x^{\mathsf{B}}|_{J^{\mathsf{B}}} = c^{\mathsf{B}} \Delta J^{\mathsf{B}}.$$

Set $v^{\mathsf{B}} = |\dot{y}^{\mathsf{B}}_0| + 1 > 0$; so $-v^{\mathsf{B}}\psi_1 \lessdot y^{\mathsf{B}} \lessdot +v^{\mathsf{B}}\psi_1$ through t^{B}_0. Then for some closed interval $I^{\mathsf{B}}_* \subseteq I^{\mathsf{B}}$ straddling t^{B}_0, for all non-trivial closed sub-intervals $J^{\mathsf{B}} \subseteq I^{\mathsf{B}}_*$ containing t^{B}_0:

$$\left|\Delta y^{\mathsf{B}}|_{J^{\mathsf{B}}}\right| < v^{\mathsf{B}} \Delta J^{\mathsf{B}}.$$

Because x^{A} is within-bounds continuous, there is a non-trivial closed sub-interval $H^{\mathsf{A}} \subseteq I^{\mathsf{A}}$ straddling t^{A}_0, where $\nu^{\mathsf{A}}(H^{\mathsf{A}}) \subseteq \nu^{\mathsf{B}}(I^{\mathsf{B}}_*)$. Hence for every sub-interval $J^{\mathsf{A}} = [t^{\mathsf{A}}_1, t^{\mathsf{A}}_2] \subseteq H$ there is a sub-interval $J^{\mathsf{B}} = [t^{\mathsf{B}}_1 : t^{\mathsf{B}}_2] \subseteq I$ (for which perhaps $t^{\mathsf{B}}_1 > t^{\mathsf{B}}_2$), where $x^{\mathsf{A}}[t^{\mathsf{A}}_1] = x^{\mathsf{B}}[t^{\mathsf{B}}_1]$ and $x^{\mathsf{A}}[t^{\mathsf{A}}_2] = x^{\mathsf{B}}[t^{\mathsf{B}}_2]$; and so:

$$\begin{aligned} \Delta x^{\mathsf{A}}|_{J^{\mathsf{A}}} &= \pm \Delta x^{\mathsf{B}}|_{J^{\mathsf{B}}}, \text{ and} \\ \Delta y^{\mathsf{A}}|_{J^{\mathsf{A}}} &= \pm \Delta y^{\mathsf{B}}|_{J^{\mathsf{B}}}. \end{aligned}$$

But, by assumption, $\dot{x}^{\mathsf{A}}_0 = 0$. So, for all $\varepsilon^{\mathsf{A}} > 0$ (thought of as small), $-\varepsilon^{\mathsf{A}}\psi_1 \lessdot x^{\mathsf{A}} \lessdot +\varepsilon^{\mathsf{A}}\psi_1$ through t^{A}_0. Hence, for some closed interval $I^{\mathsf{A}}_\varepsilon \subseteq I^{\mathsf{A}}$ straddling t^{A}_0, for all non-trivial closed sub-intervals $J^{\mathsf{A}} \subseteq I^{\mathsf{A}}_\varepsilon$ containing t^{A}_0:

$$\left|\Delta x^{\mathsf{A}}|_{J^{\mathsf{A}}}\right| < \varepsilon^{\mathsf{A}} \Delta J^{\mathsf{A}}.$$

Put $I^{\mathsf{A}}_* = H^{\mathsf{A}} \cap I^{\mathsf{A}}_\varepsilon$. For each non-trivial closed sub-interval $J^{\mathsf{A}} \subseteq I^{\mathsf{A}}_*$ containing t^{A}_0, and corresponding $J^{\mathsf{B}} \subseteq I^{\mathsf{B}}_*$:

$$\begin{aligned} \left|\Delta y^{\mathsf{A}}|_{J^{\mathsf{A}}}\right| &= \left|\Delta y^{\mathsf{B}}|_{J^{\mathsf{B}}}\right| \\ &< v^{\mathsf{B}} \Delta J^{\mathsf{B}} \\ &= \frac{v^{\mathsf{B}}}{|c^{\mathsf{B}}|} \left|\Delta x^{\mathsf{B}}|_{J^{\mathsf{B}}}\right| \\ &= \frac{v^{\mathsf{B}}}{|c^{\mathsf{B}}|} \left|\Delta x^{\mathsf{A}}|_{J^{\mathsf{A}}}\right| \\ &< \frac{v^{\mathsf{B}}\varepsilon^{\mathsf{A}}}{|c^{\mathsf{B}}|} \Delta J^{\mathsf{A}}. \end{aligned}$$

Given $\varepsilon_N, \varepsilon_P > 0$, choose $\varepsilon^{\mathsf{A}} = (|c^{\mathsf{B}}|/v^{\mathsf{B}}) \min\{\varepsilon_N, \varepsilon_P\}$ and form I^{A}_* as above. Then for each non-trivial closed sub-interval $J^{\mathsf{A}} \subseteq I^{\mathsf{A}}_*$ containing t^{A}_0:

$$-\varepsilon_N \Delta J^{\mathsf{A}} < \Delta y^{\mathsf{A}}|_{J^{\mathsf{A}}} < \varepsilon_P \Delta J^{\mathsf{A}};$$

and so:

$$-\varepsilon_N\psi_1 \lessdot y^{\mathsf{A}} \lessdot \varepsilon_P\psi_1 \quad \text{through } t_0^{\mathsf{A}}.$$

Apply the third assertion of Corollary 6.3 to conclude that y^{A} has zero unique flow velocity limit $\dot{y}_0^{\mathsf{A}} = 0$ though t_0^{A}; and trivially:

$$c^{\mathsf{B}}\,\dot{y}_0^{\mathsf{A}} = \dot{x}_0^{\mathsf{A}}\,\dot{y}_0^{\mathsf{B}},$$

as claimed. ☐

The fundamental fluxion identity is also trivial when the function f has zero derivative with respect to x^{B}, because the induced fluents y^{B} and y^{A} will both have zero unique flow velocity limits.

Theorem 8.9. *Suppose that the function* $f\colon U \to \mathbb{R}$*, fluent* $x^{\mathsf{B}} = (\mathbb{R}, U, I^{\mathsf{B}}, \nu^{\mathsf{B}})$ *and fluent* $x^{\mathsf{A}} = (\mathbb{R}, U, I^{\mathsf{A}}, \nu^{\mathsf{A}})$ *satisfy the following:*

- $U \subseteq \mathbb{R}$ *is a non-trivial open interval;*
- x^{B} *has uniform flow velocity* $c^{\mathsf{B}} \neq 0$ *on* I^{B}*;*
- $x^{\mathsf{B}}[t_0^{\mathsf{B}}] = x_0 \in U$*, where* $t_0^{\mathsf{B}} \in (I^{\mathsf{B}})^o$*;*
- f *has zero derivative with respect to* x^{B} *through* t_0^{B}*; so the induced fluent* $y^{\mathsf{B}} = f \circ x^{\mathsf{B}}$ *has zero unique flow velocity limit* $\dot{y}_0^{\mathsf{B}} \equiv \dot{y}^{\mathsf{B}}|_{t_0^{\mathsf{B}}} = 0$ *through* t_0^{B}*;*
- $x^{\mathsf{A}}[t_0^{\mathsf{A}}] = x_0$ *for some* $t_0^{\mathsf{A}} \in (I^{\mathsf{A}})^o$*;*
- *and* x^{A} *has unique flow velocity limit* $\dot{x}_0^{\mathsf{A}} \equiv \dot{x}^{\mathsf{A}}|_{t_0^{\mathsf{A}}}$ *through* t_0^{A}*.*

Write $y^{\mathsf{A}} \equiv f \circ x^{\mathsf{A}}$*. Then* y^{A} *has a zero unique flow velocity limit* $\dot{y}_0^{\mathsf{A}} \equiv \dot{y}^{\mathsf{A}}|_{t_0^{\mathsf{A}}} = 0$ *through* t_0*; and trivially:*

$$c^{\mathsf{B}}\,\dot{y}_0^{\mathsf{A}} = \dot{x}_0^{\mathsf{A}}\,\dot{y}_0^{\mathsf{B}}.$$

Proof. For each closed sub-interval $J^{\mathsf{B}} \subseteq I^{\mathsf{B}}$, fluent x^{B} satisfies:

$$\Delta x^{\mathsf{B}}|_{J^{\mathsf{B}}} = c^{\mathsf{B}}\Delta J^{\mathsf{B}}.$$

By assumption, $\dot{y}_0^{\mathsf{B}} = 0$. So, for all $\varepsilon^{\mathsf{B}} > 0$ (thought of as small), $-\varepsilon^{\mathsf{B}}\psi_1 \lessdot y^{\mathsf{B}} \lessdot +\varepsilon^{\mathsf{B}}\psi_1$ through t_0^{A}. Hence, for some closed interval $I_*^{\mathsf{B}} \subseteq I^{\mathsf{B}}$ straddling t_0^{B}, for all non-trivial closed sub-intervals $J^{\mathsf{B}} \subseteq I_*^{\mathsf{B}}$ containing t_0^{B}:

$$\left|\Delta y^{\mathsf{B}}|_{J^{\mathsf{B}}}\right| < \varepsilon^{\mathsf{B}}\Delta J^{\mathsf{B}}.$$

Because x^{A} is within-bounds continuous, there is a non-trivial closed sub-interval $H^{\mathsf{A}} \subseteq I^{\mathsf{A}}$ straddling t_0^{A}, where $\nu^{\mathsf{A}}(H^{\mathsf{A}}) \subseteq \nu^{\mathsf{B}}(I_*^{\mathsf{B}})$. Hence for every sub-interval $J^{\mathsf{A}} = [t_1^{\mathsf{A}}, t_2^{\mathsf{A}}] \subseteq H^{\mathsf{A}}$ there is a sub-interval $J^{\mathsf{B}} = [t_1^{\mathsf{B}} : t_2^{\mathsf{B}}] \subseteq I$ (for which perhaps $t_1^{\mathsf{B}} > t_2^{\mathsf{B}}$), where $x^{\mathsf{A}}[t_1^{\mathsf{A}}] = x^{\mathsf{B}}[t_1^{\mathsf{B}}]$ and $x^{\mathsf{A}}[t_2^{\mathsf{A}}] = x^{\mathsf{B}}[t_2^{\mathsf{B}}]$; and so:

$$\begin{aligned} \Delta x^{\mathsf{A}}|_{J^{\mathsf{A}}} &= \pm \Delta x^{\mathsf{B}}|_{J^{\mathsf{B}}}, \text{ and} \\ \Delta y^{\mathsf{A}}|_{J^{\mathsf{A}}} &= \pm \Delta y^{\mathsf{B}}|_{J^{\mathsf{B}}}. \end{aligned}$$

Set $u^{\mathsf{A}} = |\dot{x}_0^{\mathsf{A}}| + 1 > 0$; so $-u^{\mathsf{A}}\psi_1 \lessdot x^{\mathsf{A}} \lessdot +u^{\mathsf{A}}\psi_1$ through t_0^{A}.

Then for some closed interval $I_u^{\mathsf{A}} \subseteq I^{\mathsf{A}}$ straddling t_0^{A}, for all non-trivial closed sub-intervals $J^{\mathsf{A}} \subseteq I_u^{\mathsf{A}}$ containing t_0^{A}:

$$\left| \Delta x^{\mathsf{A}}|_{J^{\mathsf{A}}} \right| < u^{\mathsf{A}} \Delta J^{\mathsf{A}}.$$

Put $I_*^{\mathsf{A}} = H^{\mathsf{A}} \cap I_u^{\mathsf{A}}$. For each non-trivial closed sub-interval $J^{\mathsf{A}} \subseteq I_*^{\mathsf{A}}$ containing t_0^{A}, and corresponding $J^{\mathsf{B}} \subseteq I_*^{\mathsf{B}}$:

$$\begin{aligned} \left| \Delta y^{\mathsf{A}}|_{J^{\mathsf{A}}} \right| &= \left| \Delta y^{\mathsf{B}}|_{J^{\mathsf{B}}} \right| \\ &< \varepsilon^{\mathsf{B}} \Delta J^{\mathsf{B}} \\ &= \frac{\varepsilon^{\mathsf{B}}}{|c^{\mathsf{B}}|} \left| \Delta x^{\mathsf{B}}|_{J^{\mathsf{B}}} \right| \\ &= \frac{\varepsilon^{\mathsf{B}}}{|c^{\mathsf{B}}|} \left| \Delta x^{\mathsf{A}}|_{J^{\mathsf{A}}} \right| \\ &< \frac{\varepsilon^{\mathsf{B}} u^{\mathsf{A}}}{|c^{\mathsf{B}}|} \Delta J^{\mathsf{A}}. \end{aligned}$$

Given $\varepsilon_N, \varepsilon_P > 0$, choose $\varepsilon^{\mathsf{B}} = (|c^{\mathsf{B}}|/u^{\mathsf{A}}) \min\{\varepsilon_N, \varepsilon_P\}$ and form I_*^{A} as above. Then for each non-trivial closed sub-interval $J^{\mathsf{A}} \subseteq I_*^{\mathsf{A}}$ containing t_0^{A}:

$$-\varepsilon_N \Delta J^{\mathsf{A}} < \Delta y^{\mathsf{A}}|_{J^{\mathsf{A}}} < \varepsilon_P \Delta J^{\mathsf{A}};$$

and so:

$$-\varepsilon_N \psi_1 \lessdot y^{\mathsf{A}} \lessdot \varepsilon_P \psi_1 \quad \text{through } t_0^{\mathsf{A}}.$$

Apply the third assertion of Corollary 6.3 to conclude that y^{A} has zero unique flow velocity limit $\dot{y}_0^{\mathsf{A}} = 0$ though t_0^{A}; and trivially:

$$c^{\mathsf{B}}\, \dot{y}_0^{\mathsf{A}} = \dot{x}_0^{\mathsf{A}}\, \dot{y}_0^{\mathsf{B}},$$

as claimed. □

8.5.4 Final Basic FFI

We can now state the final form of the basic fundamental fluxion identity, where one of the fluents has uniform flow velocity and both fluents are eligible test fluents for differentiability$^{(\mathrm{Def\ 8.1})}$ at a point x_0.

Theorem 8.10. *Suppose that the function $f\colon U \to \mathbb{R}$, fluent $x^{\mathsf{B}} = (\mathbb{R}, U, I^{\mathsf{B}}, \nu^{\mathsf{B}})$ and fluent $x^{\mathsf{A}} = (\mathbb{R}, U, I^{\mathsf{A}}, \nu^{\mathsf{A}})$ satisfy the following:*

- *$U \subseteq \mathbb{R}$ is a non-trivial open interval;*
- *x^{B} has uniform flow velocity c^{B} on I^{B};*
- *$x^{\mathsf{B}}[t_0^{\mathsf{B}}] = x_0 \in U$, where $t_0^{\mathsf{B}} \in (I^{\mathsf{B}})^o$;*
- *f is differentiable with respect to x^{B} at x_0 through t_0^{B}; so the induced fluent $y^{\mathsf{B}} = f \circ x^{\mathsf{B}}$ has unique flow velocity limit $\dot{y}_0^{\mathsf{B}} \equiv \dot{y}^{\mathsf{B}}|_{t_0^{\mathsf{B}}}$ through t_0^{B};*
- *$x^{\mathsf{A}}[t_0^{\mathsf{A}}] = x_0$ for some $t_0^{\mathsf{A}} \in (I^{\mathsf{A}})^o$;*
- *and x^{A} has unique flow velocity limit $\dot{x}_0^{\mathsf{A}} \equiv \dot{x}^{\mathsf{A}}|_{t_0^{\mathsf{A}}}$ through t_0^{A}.*

Construct the induced fluent $y^{\mathsf{A}} \equiv f \circ x^{\mathsf{A}}$. Then y^{A} has unique flow velocity limit $\dot{y}_0^{\mathsf{A}} \equiv \dot{y}^{\mathsf{A}}|_{t_0^{\mathsf{A}}}$ through t_0^{A}, and satisfies:

$$c^{\mathsf{B}}\,\dot{y}_0^{\mathsf{A}} = \dot{x}_0^{\mathsf{A}}\,\dot{y}_0^{\mathsf{B}}.$$

Proof. This is just a modified statement of Corollary 8.7 for non-zero cases, while encompassing the zero cases of Section 8.5.3. One can easily check that between them they cover all positive, negative and zero combinations of the key quantities involved. ☐

8.6 Defining the Derivative

The fundamental fluxion identity with a uniform flow velocity base fluent implies that it's sufficient to test for differentiability with any uniform flow velocity test fluent, provided the flow velocity is non-zero. This reduces the burden of proof in establishing if a function is differentiable. It's the basis of the approach to defining the derivative which is taken here.

Theorem 8.11. *Suppose that $U \subseteq \mathbb{R}$ is a non-trivial open interval; and the fluent $x^\star = (\mathbb{R}, U, I^\star, \nu^\star)$ is a uniform flow velocity test fluent for differentiability$^{(Def\ 8.1)}$ at $x_0 \in U$ through $t_0^\star \in (I^\star)^o$, where $x^\star$ has non-zero uniform flow velocity on its indexing interval $I^\star$.*

If the function $f\colon U \to \mathbb{R}$ is differentiable with respect to $x^\star$ at $x_0 \in U$ through $t_0^\star$; then f is differentiable at x_0 (ie. is differentiable with respect to every eligible test fluent).

Proof. Suppose $f\colon U \to \mathbb{R}$ is differentiable with respect to $x^\star$ at $x_0 \in U$ through $t_0^\star$. Choose any eligible test fluent $x^{\star\star} = (\mathbb{R}, U, I^{\star\star}, \nu^{\star\star})$ for differentiability at $x_0 \in U$ through $t_0^{\star\star} \in (I^{\star\star})^o$, where $x^{\star\star}[t_0^{\star\star}] = x_0$.

Then, by Theorem 8.10, the induced fluent $f \circ x^{\star\star}$ has a unique flow velocity limit. Hence f is differentiable with respect to $x^{\star\star}$; and this is true of all such $x^{\star\star}$, as required. ☐

Isaac Newton would probably be unsurprised by the substance of the next lemma and definition. The language isn't his.

The central idea is that if x is an independent variable involved in a function $f(x)$, then we can always find a parametrisation $t^\star$ where the corresponding fluent $x^\star$ traverses an interval containing values of x of interest and where $x^\star$ has non-zero uniform flow velocity (eg. by ensuring $t^\star = x/c$ for some non-zero constant c). The next lemma ensures that the value of the resulting derivative is independent of the choice of flow velocity.

Lemma 8.12. *Suppose that $U \subseteq \mathbb{R}$ is a non-trivial open interval; and function $f\colon U \to \mathbb{R}$ is differentiable at $x_0 \in U$. Suppose also that x^{A} and x^{B} are uniform flow velocity test fluents for differentiability at x_0 through t_0^{A} and t_0^{B} respectively, where their uniform flow velocities are both non-zero.*

Then the induced fluents $y^{\mathsf{A}} = f \circ x^{\mathsf{A}}$ and $y^{\mathsf{B}} = f \circ x^{\mathsf{B}}$ have unique flow velocity limits through t_0^{A} and t_0^{B} respectively; and:

$$\frac{\dot{y}^{\mathsf{A}}}{\dot{x}^{\mathsf{A}}} = \frac{\dot{y}^{\mathsf{B}}}{\dot{x}^{\mathsf{B}}} \quad \text{through } t_0^{\mathsf{A}} \text{ and } t_0^{\mathsf{B}}.$$

Proof. Function f and fluents x^{A} and x^{B} all qualify for Theorem 8.10. Hence:

$$\dot{x}^{\mathsf{B}}\,\dot{y}^{\mathsf{A}} = \dot{x}^{\mathsf{A}}\,\dot{y}^{\mathsf{B}} \quad \text{through } t_0^{\mathsf{A}} \text{ and } t_0^{\mathsf{B}}.$$

Rearranging yields the result. ☐

Definition 8.5. Suppose that $U \subseteq \mathbb{R}$ is a non-trivial open interval; function $f\colon U \to \mathbb{R}$ is differentiable at $x_0 \in U$; and $x^\star = (\mathbb{R}, U, I^\star, \nu^\star)$ is a uniform flow velocity test fluent for differentiability at x_0 through $t_0^\star \in (I^\star)^o$ with non-zero uniform flow velocity.

Then the *derivative of f at x_0* is denoted $f'(x_0)$ and equals:

$$f'(x_0) = \frac{\dot{y}_0^\star}{\dot{x}_0^\star},$$

where $\dot{y}_0^\star = \dot{y}^\star|_{t_0^\star}$ and $\dot{x}_0^\star = \dot{x}^\star|_{t_0^\star}$. (In fact, the derived fluent $\dot{x}^\star$ is constant.)

By Lemma 8.12, $f'(x_0)$ is independent of the choice of $x^\star$.

Remark 8.3. The definition of the derivative here involves the quotient of two ordinary numbers, unlike the classical formulation as a fraction that approaches a limit, or (non-classical) methods involving a quotient of two infinitesimals.

Remark 8.4. When the indexing interval $I^\star$ is contained in the interior of a larger indexing interval, we may be able to extend the construction of unique flow velocity limits to the whole of $I^\star$, so that $\dot{x}^\star$ and $\dot{y}^\star$ are fluxions defined on $I^\star$. In that case, the derived function f' becomes a quotient $\dot{y}^\star/\dot{x}^\star$ of two fluxions on $I^\star$, which is perhaps more intuitive than working with the quotient of two unique flow velocity limits through a given single time-like instant.

We might express this in equation form as:

$$f' \circ x^\star = \frac{\dot{y}^\star}{\dot{x}^\star} \quad \text{on } I^\star.$$

Example 8.5. We can choose $I^\star \subseteq U$ straddling x_0 and $x^\star = \psi_1|_{I^\star}$; in which case, $\dot{x}_0^\star = 1$ and $f'(x_0) = \dot{y}_0^\star$. This can be seen as confirming the not unreasonable suspicion that the method of flow velocities could be applied directly to forming the derivative of an ordinary function. In that case, one could view y as being parametrised by x, with $f|_{I^\star}$ acting as the evaluation function.

We now have a strategy for establishing differentiability and finding the derivative of a function $f\colon U \to \mathbb{R}$ at $x_0 \in U$, where U is a non-trivial open interval.

1. First, find a fluent $x^\star = (\mathbb{R}, U, \nu^\star, I^\star)$ with non-zero uniform flow velocity on $I^\star$ and for which $x^\star[t_0^\star] = x_0$ for some $t_0^\star \in (I^\star)^o$.

   (This implies that $x^\star$ is a uniform flow velocity test fluent for differentiability at x_0 through $t_0^\star$.)

2. Next construct the induced fluent $y^\star = f \circ x^\star$ and show that $y^\star$ has unique flow velocity limit $\dot{y}^\star$ through $t_0^\star$ to establish differentiability.

3. Now form the quotient of the two unique flow velocity limits through $t_0^\star$, say $\dot{x}_0^\star$ and $\dot{y}_0^\star$, to determine the derivative $f'(x_0) = \dot{y}_0^\star/\dot{x}_0^\star$.

We exploit this strategy in the next section in asserting the properties of the derivative.

Example 8.6. To take a simple but instructive example, suppose $f(x) = x$. Choose any uniform velocity fluent $x^\star$ flowing across the region of interest (perhaps $x[t] = 37t - 41$, or some other more sober candidate). Then $f \circ x^\star = x^\star$, which has a unique flow velocity limit at a permissible parametric instant, so f is differentiable. Its derivative $f'(x)$ is the quotient $\dot{x}^\star/\dot{x}^\star$, which equals 1.

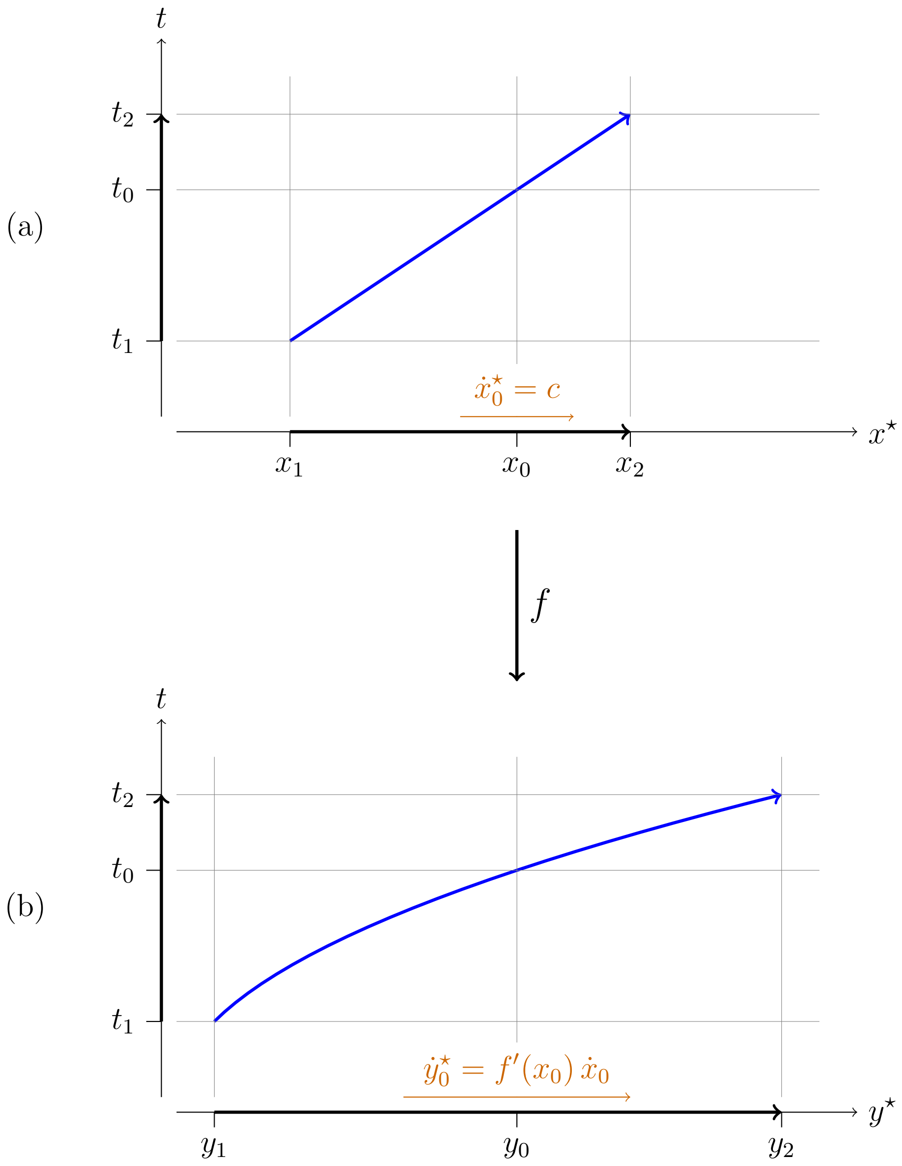

Figure 8.2: Information transfer by a differentiable function f from a non-zero uniform flow velocity fluent $x^\star$ to the induced fluent $y^\star$. As the time-like parameter t traverses an interval $[t_1, t_2]$ straddling t_0, in (a) the fluent $x^\star$ with uniform flow velocity $c > 0$ traverses the interval $[x_1, x_2]$ straddling $x_0 = x^\star[t_0]$. While this happens, in (b) the induced fluent $y^\star = f \circ x^\star$ travels from $y_1 = f(x_1)$ through $y_0 = f(x_0)$ to $y_2 = f(x_2)$. The diagram depicts $y_1 < y_0 < y_2$ and we might expect $\dot{y}_0 = \dot{y}|_{t_0} > 0$, but there is a multiplicity of possibilities in practice.

It may seem somewhat arbitrary that a uniform flow velocity fluent $x^\star = (\mathbb{R}, U, I^\star, \nu^\star)$ is sufficient to prove differentiability. We can think of $x^\star$ as flowing across $x_0 = x^\star[t_0^\star]$ as the time-like parameter $t^\star$ traverses the indexing interval $I^\star$, or a non-trivial closed sub-interval of it straddling $t_0^\star$. As $t^\star$ progresses, fluent $x^\star$ itself will traverse a closed sub-interval of U which straddles x_0. The induced fluent $y^\star$ is the image under f of this motion.

The fluents $x^\star$ and $y^\star$ are therefore capturing information in the vicinity of x_0 about how the image under f behaves. Expressed informally, we've shown in Lemma 8.12 that the faster $x^\star$ flows across x_0, the faster $y^\star$ travels in $f(U)$, such that their unique flow velocity limits are always in the same proportion, independently of parametrisation. (Although $y^\star$ passes through $f(x_0)$, it may not flow across $f(x_0)$, from one side to the other, unless either $y^\star \lessdot 0$ or $y^\star \gtrdot 0$ through $t_0^\star$.)

Figure 8.2 illustrates these ideas for a case where $f(x) \propto x^2$.

The derivative opens up a number of variations on the fundamental fluxion identity, including the following theorem, which applies to any appropriate eligible test fluent for differentiability. It shows that, when a function is differentiable, we can choose any parametrisation that has a non-zero unique flow velocity limit in evaluating the derivative.

Theorem 8.13. *Suppose that $U \subseteq \mathbb{R}$ is a non-trivial open interval; function $f\colon U \to \mathbb{R}$ is differentiable at $x_0 \in U$; that $x^\star$ is an eligible test fluent for differentiability at x_0 through $t_0^\star$; and that $y^\star = f \circ x^\star$. Then:*

$$\dot{y}^\star = f'(x_0)\,\dot{x}^\star \quad \textit{through } t_0^\star.$$

Hence, if $\dot{x}^\star$ through $t_0^\star$ is non-zero, then:

$$f'(x_0) = \frac{\dot{y}^\star}{\dot{x}^\star} \quad \textit{taken through } t_0^\star.$$

Proof. Construct a closed interval $K \subseteq U$ straddling x_0; and choose a flow velocity $c^{\mathsf{B}} \neq 0$. There is a fluent $x^{\mathsf{B}} = (\mathbb{R}, U, I^{\mathsf{B}}, \nu^{\mathsf{B}})$ traversing K with uniform flow velocity c^{B}, where $x^{\mathsf{B}}[t_0^{\mathsf{B}}] = x_0$ for some $t_0^{\mathsf{B}} \in (I^{\mathsf{B}})^o$ and $c^{\mathsf{B}} = \dot{x}^{\mathsf{B}}$ through t_0^{B}. (There are many such.)

Then x^{B} and $x^\star$ satisfy the requirements of Theorem 8.10 (with $x^{\mathsf{A}} \mapsto x^\star$); hence $\dot{x}^{\mathsf{B}}\,\dot{y}^\star = \dot{x}^\star\,\dot{y}^{\mathsf{B}}$ through $t_0^\star$ and t_0^{B}.

Noting that, by Definition 8.5, $f'(x_0) = \dot{y}^{\mathsf{B}}/\dot{x}^{\mathsf{B}}$ through t_0^{B} and rearranging reproduces the required results. $\square$

This theorem leads to a more general form of the fundamental fluxion identity, which applies to any pair of eligible test fluents for differentiability at the same point x_0. The next corollary contains the details.

Corollary 8.14. *Suppose that $U \subseteq \mathbb{R}$ is a non-trivial open interval; function $f\colon U \to \mathbb{R}$ is differentiable at $x_0 \in U$; and that x^{A} and x^{B} are eligible test fluents for differentiability at x_0 through t^{A}_0 and t^{B}_0 respectively.*

Write $\dot{x}^{\mathsf{A}}_0 \equiv \dot{x}^{\mathsf{A}}|_{t^{\mathsf{A}}_0}$, $\dot{x}^{\mathsf{B}}_0 \equiv \dot{x}^{\mathsf{B}}|_{t^{\mathsf{B}}_0}$, $\dot{y}^{\mathsf{A}}_0 \equiv \dot{y}^{\mathsf{A}}|_{t^{\mathsf{A}}_0}$ and $\dot{y}^{\mathsf{B}}_0 \equiv \dot{x}^{\mathsf{B}}|_{t^{\mathsf{B}}_0}$. Then:

$$\dot{x}^{\mathsf{B}}_0\,\dot{y}^{\mathsf{A}}_0 = \dot{x}^{\mathsf{A}}_0\,\dot{y}^{\mathsf{B}}_0.$$

Proof. Applying Theorem 8.13:

$$\begin{aligned}\dot{y}^{\mathsf{A}}_0 &= f'(x_0)\,\dot{x}^{\mathsf{A}}_0\\ \dot{y}^{\mathsf{B}}_0 &= f'(x_0)\,\dot{x}^{\mathsf{B}}_0\end{aligned}$$

If $f'(x_0) = 0$, then $\dot{y}^{\mathsf{A}}_0 = \dot{y}^{\mathsf{B}}_0 = 0$ and the result is trivial; so assume $f'(x_0) \neq 0$. But then:

$$f'(x_0)\,\dot{x}^{\mathsf{B}}_0\,\dot{y}^{\mathsf{A}}_0 = f'(x_0)\,\dot{x}^{\mathsf{A}}_0\,\dot{y}^{\mathsf{B}}_0$$

Dividing by $f'(x_0)$ gives the desired result. ☐

Before moving on, we may wish to confirm that the sign of the derivative in Definition 8.3 is consistent with the actual value of the derivative. Here is the necessary assurance when the eligible test fluent for differentiability involved has non-zero unique flow velocity limit.

Corollary 8.15. *Suppose $U \subseteq \mathbb{R}$ is a non-trivial open interval; function $f\colon U \to \mathbb{R}$ is differentiable at $x_0 \in U$; and $x^\star$ is an eligible test fluent for differentiability at x_0 through $t^\star_0$ with non-zero unique flow velocity limit. Then:*

- *f has negative derivative at x_0 with respect to $x^\star$ through $t^\star_0$ iff $f'(x_0) < 0$;*
- *f has positive derivative at x_0 with respect to $x^\star$ through $t^\star_0$ iff $f'(x_0) > 0$;*
- *f has zero derivative at x_0 with respect to $x^\star$ through $t^\star_0$ iff $f'(x_0) = 0$.*

Proof. Apply Theorem 8.13 to each of the cases in Definition 8.3. ☐

8.7 Properties

The basic properties of derivatives echo those of fluxions in the previous chapter. There is a slight awkwardness here, because a derivative is defined at a given point (usually denoted x_0 in this document), while a derived function might exist on the whole of the original function domain U. Both situations yield similar results, eg. the product rule can be formulated for a derivative defined at x_0 or analogously for a derived function on the whole of U.

Theorem 8.16. *Suppose $a \in \mathbb{R}$; and that f, g and h are three real-valued functions $U \to \mathbb{R}$ defined on the same non-trivial open interval U. Assume also that the derivatives f' and g' are well-defined on U (ie. both exist on U). Then the following basic relations also hold.*

1. *(Translational invariance) If function $h = f + a$, then h' exists and $h' = f'$.*
2. *(Scaling) If function $h = a\,f$, then h' exists and $h' = a\,f'$.*
3. *(Summation) If function $h = f + g$, then h' exists and $h' = f' + g'$.*
4. *(Product Rule) If function $h = fg$, then h' exists and $h' = f'g + fg'$.*

Proof. We proceed by rewriting the derivatives of the functions involved as a quotient of fluxions.

Fix $x_0 \in U$ and let $x^\star = (\mathbb{R}, U, I^\star, \nu^\star)$ be a uniform flow velocity test fluent for differentiability at x_0 through $t_0^\star \in (I^\star)^o$; so $x^\star[t_0^\star] = x_0$. We also require $\dot{x}^\star \neq 0$ through $t_0^\star$. We may be tempted to choose $x^\star = \psi_1|_{I^\star}$ (with $I^\star \subseteq U$ straddling x_0), but it's conceptually more informative to remain agnostic about the exact nature of $x^\star$.

In what follows, label the induced fluents as:

$$\begin{aligned} y^{\mathsf{F}} &= f \circ x^\star \\ y^{\mathsf{G}} &= g \circ x^\star \\ y^{\mathsf{H}} &= h \circ x^\star \end{aligned}$$

By assumption, f and g are differentiable at x_0, so there are unique flow velocity limits $\dot{y}^{\mathsf{F}}$ and $\dot{y}^{\mathsf{G}}$ through $t_0^\star$; and from Definition 8.5:

$$\begin{aligned} f'(x_0) &= \frac{\dot{y}^{\mathsf{F}}}{\dot{x}^\star} \quad \text{through } t_0^\star \\ g'(x_0) &= \frac{\dot{y}^{\mathsf{G}}}{\dot{x}^\star} \quad \text{through } t_0^\star \end{aligned}$$

Each assertion can now be proved by exploiting the corresponding result in Theorem 7.1 for fluxions.

For example, to prove the summation result at x_0, suppose $h = f + g$. Then $h \circ x^\star = f \circ x^\star + g \circ x^\star$, ie. fluent $y^{\mathsf{H}} = y^{\mathsf{F}} + y^{\mathsf{G}}$.

Therefore, from Theorem 7.1, the fluxion $\dot{y}^{\mathsf{H}}$ exists and $\dot{y}^{\mathsf{H}} = \dot{y}^{\mathsf{F}} + \dot{y}^{\mathsf{G}}$; so f is differentiable at x_0 and, in line with Definition 8.2, h has derivative:

$$\begin{aligned} h'(x_0) &= \frac{\dot{y}^{\mathsf{H}}}{\dot{x}^\star} \quad \text{through } t_0^\star \\ &= \frac{\dot{y}^{\mathsf{F}} + \dot{y}^{\mathsf{G}}}{\dot{x}^\star} \quad \text{through } t_0^\star \\ &= \frac{\dot{y}^{\mathsf{F}}}{\dot{x}^\star} + \frac{\dot{y}^{\mathsf{G}}}{\dot{x}^\star} \quad \text{through } t_0^\star \\ &= f'(x_0) + g'(x_0). \end{aligned}$$

This is true for all $x_0 \in U$; hence $h' = f' + g'$ as required.

It is only slightly more complex to prove the product rule at x_0. Suppose $h = fg$. Then $h \circ x^\star = (f \circ x^\star)(g \circ x^\star)$, ie. fluent $y^{\mathsf{H}} = y^{\mathsf{F}} y^{\mathsf{G}}$.

Therefore, from Theorem 7.1, the fluxion $\dot{y}^{\mathsf{H}}$ exists and $\dot{y}^{\mathsf{H}} = \dot{y}^{\mathsf{F}} y^{\mathsf{G}} + y^{\mathsf{F}} \dot{y}^{\mathsf{G}}$; so h is differentiable at x_0 and has derivative:

$$\begin{aligned} h'(x_0) &= \frac{\dot{y}^{\mathsf{H}}}{\dot{x}^\star} \quad \text{through } t_0^\star \\ &= \frac{\dot{y}^{\mathsf{F}} y^{\mathsf{G}} + y^{\mathsf{F}} \dot{y}^{\mathsf{G}}}{\dot{x}^\star} \quad \text{through } t_0^\star \\ &= \frac{\dot{y}^{\mathsf{F}}}{\dot{x}^\star} y^{\mathsf{G}} + y^{\mathsf{F}} \frac{\dot{y}^{\mathsf{G}}}{\dot{x}^\star} \quad \text{through } t_0^\star \\ &= f'(x_0)\, g(x_0) + f(x_0)\, g'(x_0). \end{aligned}$$

This is true for all $x_0 \in U$; hence $h' = f'g + fg'$ as required.

The remaining assertions can be proven similarly. ☐

Defining the derivative in terms of fluxions delivers the chain rule at little extra cost.

Theorem 8.17. *Suppose that U and V are two non-trivial open intervals, that $f\colon U \to V$ and $g\colon V \to \mathbb{R}$ are two real-valued functions, and $h = g \circ f$ is the function composition of g and f. Assume also that the derivatives $f'(x_0)$ and $g'(y_0)$ are well-defined for all $x_0 \in U$ and $y_0 \in V$.*

Then the function $h\colon U \to \mathbb{R}$ is differentiable and for all $x_0 \in U$:

$$h'(x_0) = g'(y_0)\, f'(x_0), \quad \text{where } y_0 = f(x_0).$$

Proof. Fix $x_0 \in U$ and let $x^\star = (\mathbb{R}, U, I^\star, \nu^\star)$ be a uniform flow velocity test fluent for differentiability of f at x_0 through $t_0^\star \in (I^\star)^o$; so $x^\star[t_0^\star] = x_0$. Also ensure that $\dot{x}^\star \neq 0$ through $t_0^\star$.

Label the induced fluents:

$$\begin{aligned} y^{\mathsf{F}} &= f \circ x^\star \\ z^{\mathsf{H}} &= h \circ x^\star \end{aligned}$$

By assumption, f is differentiable at x_0, so y^{F} has unique flow velocity limit $\dot{y}^{\mathsf{F}}$ through $t_0^\star$; and:

$$\dot{y}^{\mathsf{F}} = f'(x_0)\,\dot{x}^\star \quad \text{through } t_0^\star.$$

Because y^{F} has unique flow velocity limit through $t_0^\star$, it is an eligible test fluent for differentiability of g at $y_0 = f(x_0) = y^{\mathsf{F}}[t_0]$.

But $z^{\mathsf{H}} = g \circ y^{\mathsf{F}}$, where g is differentiable. Hence z^{H} also has unique flow velocity limit through $t_0^\star$; so h is differentiable with respect to $x^\star$, since $z^{\mathsf{H}} = h \circ x^\star$. So h is differentiable at x_0 and has a derivative $h'(x_0) = \dot{z}^{\mathsf{H}}/\dot{x}^\star$ through t_0.

By Theorem 8.13:

$$\dot{z}^{\mathsf{H}} = g'(y_0)\,\dot{y}^{\mathsf{F}} \quad \text{through } t_0^\star.$$

Combining the two central equations above:

$$\dot{z}^{\mathsf{H}} = g'(y_0)\,f'(x_0)\,\dot{x}^\star \quad \text{through } t_0^\star.$$

Hence $h'(x_0) = g'(y_0)\,f'(x_0)$ as claimed. □

8.8 Equivalence with the Classical Derivative

Differentiation in terms of fluxions is equivalent to the classical formulation, which can be expressed arithmetically as follows.

Definition 8.6. Given a non-trivial open interval $U \subseteq \mathbb{R}$ and the function $f\colon U \to \mathbb{R}$, we say f is *classically differentiable* at $x_0 \in U$ iff there exists $m \in \mathbb{R}$ where, for all $\varepsilon > 0$, there exists $\delta > 0$ for which for all h with $0 < |h| < \delta$:

$$\left| \frac{f(x_0 + h) - f(x_0)}{h} - m \right| < \varepsilon.$$

The value m is the *classical derivative* of f at x_0.

We use Leibniz' notation to distinguish the classical derivative of $f(x)$, writing $m = \dfrac{df}{dx}$ at x_0.

We show first that if a function is differentiable with respect to all eligible test fluents, then it is differentiable classically in the sense of Definition 8.6. It's sufficient to prove the result for just one of the many possible test fluents, and we simplify by employing the fluent continuation ψ_1, treating U as a time-like domain.

Theorem 8.18. *Suppose $U \subseteq \mathbb{R}$ is a non-trivial open interval and $f\colon U \to \mathbb{R}$ is a function on U. Define an induced fluent $y^\star = f \circ \psi_1$ on an indexing interval*

$I^\star \subseteq U$. If the fluxion $\dot{y}^\star$ is defined on $I^\star$, then f is classically differentiable at each $t_0 \in (I^\star)^o$ and has a classical derivative:

$$\frac{df}{dt} = \dot{y}^\star$$

on $(I^\star)^o$.

Proof. Fix $t_0 \in (I^\star)^o$. We show that f is classically differentiable at t_0 with the classical derivative $m = \dot{y}_0^\star = \dot{y}^\star|_{t_0}$.

Choose $\varepsilon > 0$ and set $u = \dot{y}_0^\star - \varepsilon$ and $v = \dot{y}_0^\star + \varepsilon$. Form $H^\star \subseteq I^\star$ straddling t_0, where $u\psi_1 \lessdot y^\star \lessdot v\psi_1$ on $H^\star$ through t_0. We can then set $\delta = \min\{t_0 - \min H^\star, \max H^\star - t_0\} > 0$.

Now, for $0 < h < \delta$, form the sub-interval $J_2 = [t_0, t_0 + h] \subseteq H^\star$. Then $y^\star \lessdot v\psi_1$ implies $\Delta y^\star|_{J_2} < v\Delta J_2 = (\dot{y}_0^\star + \varepsilon)h$. Noting that $\Delta y^\star|_{J_2} = f(t_0+h) - f(t_0)$ and rearranging:

$$\frac{f(t_0 + h) - f(t_0)}{h} - \dot{y}_0^\star < \varepsilon.$$

When instead $-\delta < h < 0$, put $J_1 = [t_0 + h, t] \subseteq H^\star$; and now $\Delta y^\star|_{J_1} < v\Delta J_1 = (\dot{y}_0^\star + \varepsilon)|h|$. Rewriting, $f(t_0) - f(t_0 + h) < (\dot{y}_0^\star + \varepsilon)|h|$; or:

$$\frac{f(t_0) - f(t_0 + h)}{|h|} - \dot{y}_0^\star < \varepsilon.$$

But for this case $|h| = -h$, so again:

$$\frac{f(t_0 + h) - f(t_0)}{h} - \dot{y}_0^\star < \varepsilon.$$

We've shown that this inequality holds for all h with $0 < |h| < \delta$.

A dual argument applies from $u\psi_1 \lessdot y^\star$ through t_0, leading to $u\Delta J_1 < \Delta y^\star|_{J_1}$ and $u\Delta J_2 < \Delta y^\star|_{J_2}$; from which:

$$-\varepsilon < \frac{f(t_0 + h) - f(t_0)}{h} - \dot{y}_0^\star.$$

Bringing this all together shows that $m = \dot{y}_0^\star$ satisfies the conditions of Definition 8.6 and f is classically differentiable at $t_0 \in (I^\star)^o$ with derivative equal to $\dot{y}_0^\star$, as required. ☐

The converse result asserts that if a function is differentiable classically, then it is differentiable with respect to all eligible test fluents. By Theorem 8.11 it suffices to prove this for a uniform flow velocity test fluent for differentiability at the point of interest. Because we can always be sure to find a valid candidate, the following theorem suffices.

Theorem 8.19. *Suppose $U \subseteq \mathbb{R}$ is a non-trivial open interval and assume that function $f\colon U \to \mathbb{R}$ is classically differentiable at $x_0 \in U$. Suppose also that $x^\star = (\mathbb{R}, U, I^\star, \nu^\star)$ is a uniform flow velocity test fluent for differentiability at $x_0 \in U$ with non-zero flow velocity on $I^\star$.*

Then f is differentiable at x_0 with respect to $x^\star$; and the two derivatives are equal:

$$f'(x_0) = \frac{df}{dx}(x_0).$$

Proof. Assume f is classically differentiable at x_0 with derivative $m = \dfrac{df}{dx}(x_0)$ and that $x^\star$ has non-zero uniform flow velocity $\dot{x}_0^\star = c \neq 0$.

Put $x_0 = x \star [t_0] \in U$, where $t_0 \in (I^\star)^o$. We wish to show the induced fluent $y^\star = f \circ x^\star$ has unique flow velocity limit through t_0 and $\dot{y}_0^\star = m\dot{x}_0^\star$.

Choose any $v > m\dot{x}_0^\star$ and define $\varepsilon > 0$ so that $\varepsilon|c| = v - m\dot{x}_0^\star = v - mc$. Then there is a $\delta > 0$ for which for all h with $0 < |h| < \delta$:

$$\left| \frac{f(x_0 + h) - f(x_0)}{h} - m \right| < \varepsilon,$$

which implies $mh - \varepsilon|h| < f(x_0 + h) - f(x_0) < mh + \varepsilon|h|$.

Set interval $I_2 = [t_0, t_0 + \frac{1}{2|c|}\delta]$ and select any sub-interval $J_2 = [t_0, t_2] \subseteq I_2$ containing t_0. Note that $|c|(t_2 - t_1) \leq \frac{1}{2}\delta < \delta$ and $x^\star[t_2] = x_0 + c(t_2 - t_1)$.

Hence:

$$\begin{aligned}
\Delta y^\star|_{J_2} &= y^\star[t_2] - y^\star[t_0] \\
&= f(x_0 + c(t_2 - t_1)) - f(x_0) \\
&< (mc + \varepsilon|c|)(t_2 - t_1) \\
&= v\Delta J_2.
\end{aligned}$$

We can form $I_2 = [t_0 - \frac{1}{2|c|}\delta, t_0]$ similarly and take any sub-interval $J_1 = [t_1, t_0] \subseteq I_1$. By a similar argument, $\Delta y^\star|_{J_1} < v\Delta J_1$.

Now put $H = I_1 \cup I_2$. Any non-trivial closed sub-interval $J \subseteq H$ containing t_0 can be decomposed into sub-intervals J_1 and J_2 as above; and the interim results combine to ensure $\Delta y^\star|_J < v\Delta J$. This is true for all such $J \subseteq H$, so $y^\star \lessdot v\psi_1$ through t_0 for all $v > m\dot{x}_0^\star$.

A dual argument with $u < m\dot{x}_0^\star$ shows that also $u\psi_1 \lessdot y^\star$ for all such u.

Applying Corollary 6.3, then $y^\star$ has unique flow velocity limit and $\grave{y}^\star = \acute{y}^\star = \dot{y}^\star = m\dot{x}_0$. Hence f is differentiable at x_0 with respect to $x^\star$; and has the same derivative m there as the classical result. $\square$

Part IV

Appendices

Bibliography

[1] *Collins Dictionary of Mathematics*

[2] Isaac Newton (translated by John Harris) *Introduction to the Quadrature of Curves*

[3] Isaac Newton (translated by John Colson) *The Method of Fluxions and Infinite Series*

[4] *Banach space* on Wikipedia
`https://en.wikipedia.org/wiki/Banach_space`

[5] *Extended real number line* on Wikipedia
`https://en.wikipedia.org/wiki/Extended_real_number_line`

[6] *Extreme value theorem* on Wikipedia
`https://en.wikipedia.org/wiki/Extreme_value_theorem`

[7] *Heaviside step function* on Wikipedia
`https://en.wikipedia.org/wiki/Heaviside_step_function`

[8] *Interval (mathematics)* on Wikipedia
`https://en.wikipedia.org/wiki/Interval_(mathematics)`

[9] *Lipschitz continuity* on Wikipedia
`https://en.wikipedia.org/wiki/Lipschitz_continuity`

[10] *Logarithmic scale* on Wikipedia
`https://en.wikipedia.org/wiki/Logarithmic_scale`

[11] *Mean value theorem* on Wikipedia
`https://en.wikipedia.org/wiki/Mean_value_theorem`

[12] *Real number* on Wikipedia
`https://en.wikipedia.org/wiki/Real_number`

[13] *Rolle's theorem* on Wikipedia
`https://en.wikipedia.org/wiki/Rolle's_theorem`

[14] *Set-builder notation* on Wikipedia
`https://en.wikipedia.org/wiki/Set-builder_notation`

[15] *Square root of 2* on Wikipedia
`https://en.wikipedia.org/wiki/Square_root_of_2`

[16] *Topological space* on Wikipedia
`https://en.wikipedia.org/wiki/Topological_space`

[17] *Upper set* on Wikipedia
`https://en.wikipedia.org/wiki/Upper_set`

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